Practice Problems in Preparation for Calculus Final Exam

PROBLEM 1

Consider $r(x) = \dfrac{x^2 - x - 12}{x + 3}$.

A) Evaluate $\lim\limits_{x \to -3} r(x)$.

B) Show that the function does not have a horizontal asymptote.

PROBLEM 2[♥]

Consider $y(x) = \dfrac{1}{x}$. Use the limit definition of the derivative to find the slope of any line tangent to *y*(*x*).

PROBLEM 3[♠]

Verify by differentiation $\int e^{3x} \cdot \left(\dfrac{1}{x} + 3\ln(x) \right) = e^{3x}\left(\ln(x)\right)$.

PROBLEM 4[♣]

Consider $f(x) = x^3 + x^2 - x - 1$.

A) Label any intercepts, relative extrema, absolute extrema, and inflection points.
B) Name the intervals of increasing and decreasing behavior as well as the intervals of concavity.
C) Use calculus to sketch a graph of *f*(*x*).
D) Determine the area between *f*(*x*) and the *x*-axis along the interval [–2,2].

PROBLEM 5[♦]

State and illustrate the Intermediate Value Theorem.

PROBLEM 6

Decide which is the largest, $\int_0^2 xe^{x^2}\,dx$, $\int_{\pi/6}^{\pi/2} \cos(x)\,dx$, or $\int_0^{1} 6xe^{3x}\,dx$.

♥ The student should be able to do this problem with a variety of functions, e.g., $f(x) = \sqrt{x+1}$ and $p(x) = 2x - x^2$.

♠ The student should be able to do this problem with a variety of integration facts, e.g., $\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C$.

♣ The student should be able to do this problem with a variety of functions, e.g., $f(x) = e^{-x^2}$ and $g(x) = \dfrac{\ln(x)}{\sqrt{x}}$.

♦ The student should be able to do this problem with a variety of theorems, e.g., the Mean Value Theorem.

SOLUTION to PROBLEM 1

A) Evaluate $\lim_{x\to -3} r(x)$.

$$\lim_{x\to -3}\frac{x^2-x-12}{x+3}=\lim_{x\to -3}\frac{(x-4)\cancel{(x+3)}}{\cancel{x+3}}=\lim_{x\to -3} x-4=-3-4=-7$$

B) Evaluate $\lim_{x\to\infty} r(x)$.

$$\lim_{x\to\infty}\left[\frac{x^2-x-12}{x+3}\right]=\lim_{x\to\infty}\left[\frac{x^2-x-12}{x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\right]=\lim_{x\to\infty}\left[\frac{\frac{x^2}{x}-\frac{x}{x}-\frac{12}{x}}{\frac{x}{x}+\frac{3}{x}}\right]=\lim_{x\to\infty}\left[\frac{x-1-\frac{12}{x}}{1+\frac{3}{x}}\right]=\frac{\infty-1-\frac{12}{\infty}}{1+\frac{3}{\infty}}=\frac{\infty-1}{1}=\infty$$

$$\lim_{x\to-\infty}\left[\frac{x^2-x-12}{x+3}\right]=\lim_{x\to-\infty}\left[\frac{x^2-x-12}{x+3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\right]=\lim_{x\to-\infty}\left[\frac{\frac{x^2}{x}-\frac{x}{x}-\frac{12}{x}}{\frac{x}{x}+\frac{3}{x}}\right]=\lim_{x\to-\infty}\left[\frac{x-1-\frac{12}{x}}{1+\frac{3}{x}}\right]=\frac{-\infty-1-\frac{12}{-\infty}}{1+\frac{3}{-\infty}}=\frac{-\infty-1}{1}=-\infty$$

SOLUTION to PROBLEM 2

$$m=\lim_{h\to0}\left[\frac{y(x+h)-y(x)}{h}\right]=\lim_{h\to0}\left[\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\right]=\lim_{h\to0}\left[\frac{\frac{1}{x+h}\cdot\frac{x}{x}-\frac{1}{x}\cdot\frac{x+h}{x+h}}{h}\right]$$

$$m=\lim_{h\to0}\left[\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}\right]=\lim_{h\to0}\left[\frac{\frac{x-(x+h)}{x(x+h)}}{h}\right]=\lim_{h\to0}\left[\frac{\frac{-h}{x(x+h)}}{h}\right]$$

$$m=\lim_{h\to0}\left[\frac{-h}{x(x+h)}\div h\right]=\lim_{h\to0}\left[\frac{-\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}\right]=\lim_{h\to0}\left[\frac{-1}{x(x+h)}\right]$$

$$m=\frac{-1}{x(x+0)}=-\frac{1}{x^2}$$

SOLUTION to PROBLEM 3

$$F(x) = e^{3x}(\ln(x))$$

$$F'(x) = e^{3x} \cdot \frac{d}{dx}[\ln(x)] + \ln(x) \cdot \frac{d}{dx}[e^{3x}]$$

$$F'(x) = e^{3x} \cdot \frac{1}{x} + \ln(x) \cdot 3e^{3x}$$

$$F'(x) = e^{3x} \cdot \frac{1}{x} + 3e^{3x} \cdot \ln(x)$$

$$F'(x) = e^{3x} \cdot \left(\frac{1}{x} + 3\ln(x)\right)$$

SOLUTION to PROBLEM 4

Part A & B:

Find the roots:

$$x^3 + x^2 - x - 1 = 0$$

$$x^2(x+1) - 1(x+1) = 0$$

$$(x^2 - 1)(x+1) = 0$$

$$(x+1)(x-1)(x+1) = 0$$

$$x = 1, -1$$

$(1,0)$ and $(-1,0)$

Find the first derivative:

$$f'(x) = 3x^2 + 2x - 1$$

Find the critical numbers of *f*, the *x*-values where f' is undefined or equal to zero:

$$3x^2 + 2x - 1 = 0$$

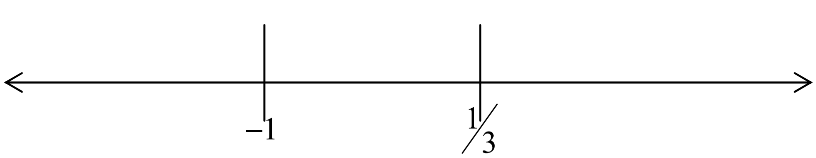
$$(3x-1)(x+1) = 0$$

$$3x - 1 = 0 \quad x + 1 = 0$$

$$x = \frac{1}{3}, \quad -1$$

Test on all sides of the critical numbers and draw conclusions according to Theorem 1:

f ' *POSITIVE* *NEGATIVE* *POSITIVE*



f *INCREASING* *DECREASING* *INCREASING*

State the intervals of behavior: The function increases along $(-\infty,-1)\cup(1/3,\infty)$ and decreases along $(-1,1/3)$.

Find the extrema: Relative (also called local) extrema occur where the function changes from increasing to decreasing behavior along an interval of continuity or vice versa. The point $f(-1)=0$ is a relative maximum while $f(1/3)=-32/27$ is a relative minimum.

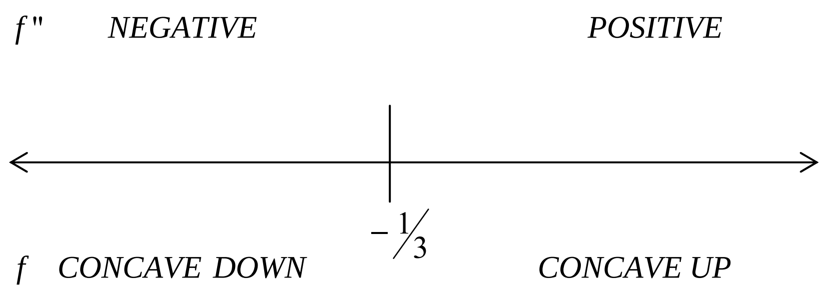
Find the second derivative:

$$f''(x)=6x+2$$

Find the critical numbers of f', the x-values where f'' is undefined or equal to zero:

$$6x+2=0$$
$$x=-\frac{1}{3}$$

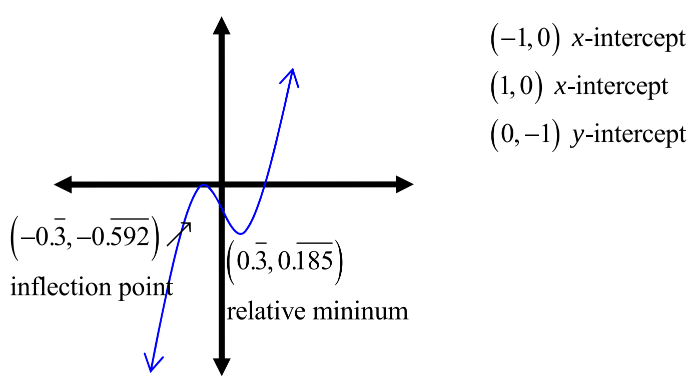
Test on all sides of the critical numbers and draw conclusions according to Theorem 2:



State the intervals of concavity: The function is concave down along $(-\infty,-1/3)$ and concave up along $(-1/3,\infty)$.

Find the inflection: Inflection points occur where the function changes in concavity along an interval of continuity. The point $f(-1/3)=-16/27$ is an inflection point.

Part C:



Part D:

$$\int_1^2 \left(x^3+x^2-x-1\right)dx-\int_{-2}^1\left(x^3+x^2-x-1\right)dx$$
$$=\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2-x\Big|_1^2-\left(\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2-x\Big|_{-2}^1\right)$$
$$=\frac{1}{4}(2)^4+\frac{1}{3}(2)^3-\frac{1}{2}(2)^2-(2)-\left(\frac{1}{4}(1)^4+\frac{1}{3}(1)^3-\frac{1}{2}(1)^2-(1)\right)-\left(\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2-x\Big|_{-2}^1\right)$$
$$=\frac{43}{12}-\left(\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2-x\Big|_{-2}^1\right)$$
$$=\frac{43}{12}-\left(\frac{1}{4}(1)^4+\frac{1}{3}(1)^3-\frac{1}{2}(1)^2-(1)-\left(\frac{1}{4}(-2)^4+\frac{1}{3}(-2)^3-\frac{1}{2}(-2)^2-(-2)\right)\right)$$
$$=\frac{43}{12}-\left(-\frac{9}{4}\right)$$
$$=\frac{35}{6}\text{ sq. units}$$

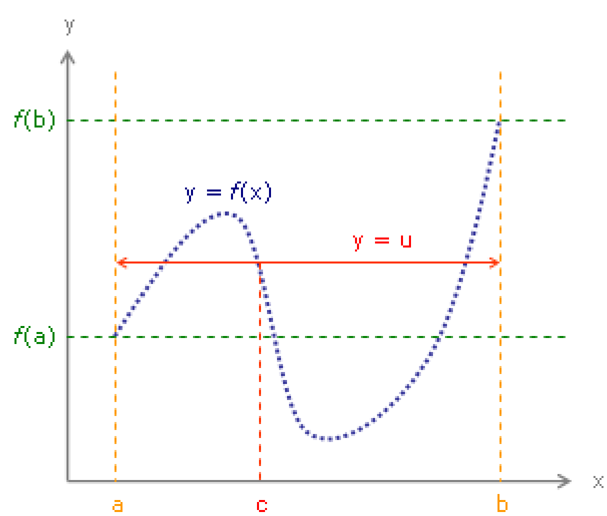
SOLUTION to PROBLEM 5

Let $f:[a,b]\to\mathbb{R}$. Suppose f is continuous and $f(a)\neq f(b)$. The Intermediate Value Theorem is below.

If $f(a)\le M\le f(b)$ or $f(b)\le M\le f(a)$, then $\exists\ c\in[a,b]\ \ni\ f(c)=M$.

In other words, the Intermediate Value Theorem states that if f is a real-valued function defined *and continuous* on the domain $[a,b]$ and *M* is a number between $f(a)$ and $f(b)$, then there exists *at least* one domain value that f maps to *M*.

In the illustration below, u plays the role of M. The illustration actually shows three possible c-values (see the three intersections between f and the line $y=u$) but labels only one.



SOLUTION to PROBLEM 6

Evaluate $\int_0^2 xe^{x^2}\,dx$

Let $u = x^2$ If $x = 0$, $u = 0$, and if $x = 2$, $u = 4$.

$du = 2x\,dx$

$\tfrac{1}{2}du = x\,dx$

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}e^4 - \frac{1}{2}e^0 = \frac{1}{2}e^4 - \frac{1}{2}\cdot 1 = \frac{1}{2}e^4 - \frac{1}{2}$$

Evaluate $\int_{\pi/6}^{\pi/2} \cos(x)\,dx$

$$\int_{\pi/6}^{\pi/2} \cos(x)\,dx = \sin(x)\Big|_{\pi/6}^{\pi/2} = \sin\left(\pi/2\right) - \left(\sin\left(\pi/6\right)\right) = 1 - \left(\frac{1}{2}\right) = \frac{1}{2}$$

Unit circle time!

Evaluate using integration by parts: $\int_0^{1`} 6xe^{3x}\,dx$

Let $u = 6x$ and $dv = e^{3x}\,dx$.

then $du = 6\,dx$ and $v = \tfrac{1}{3}e^{3x}$.

$$\text{Let}\quad I = \int 6xe^{3x}dx$$

$$\text{then}\quad I = 2xe^{3x} - \int 2e^{3x}dx$$

$$I = 2xe^{3x} - 2\int e^{3x}dx$$

$$I = 2xe^{3x} - 2\cdot\frac{1}{3}e^{3x} + C$$

$$I = \frac{2}{3}e^{3x}(3x-1) + C$$

$$\int_0^{1`} 6xe^{3x}\,dx = \frac{2}{3}e^{3(1)}\left(3(1)-1\right) - \left[\frac{2}{3}e^{3(0)}\left(3(0)-1\right)\right]$$

$$= \frac{2}{3}e^3(2) - \left[\frac{2}{3}e^0(-1)\right]$$

$$= \frac{4}{3}e^3 + \frac{2}{3}$$