Differentiation: Chain Rule

Consider the function $p(x) = f(g(x))$ where $f(x) = x^4$ and $g(x) = 7 - 3x^2$. Finding the derivatives of *f* and *g* are simple applications of the power rule and the linearity of differentiation. Finding $p'(x)$, however, looks more daunting. Fortunately, there is the *chain rule*, a rule stated below that governs differentiation of composition functions like *p*(*x*).

> *Chain Rule*: If *g* is differentiable at *x* and *f* is differentiable at *g*(*x*), then the composite function $C = f \circ g$ defined by $C(x) = f(g(x))$ is differentiable at *x* and C' is given by the product
>
> $$C'(x) = f'(g(x)) \cdot g'(x).$$

Returning to $p(x) = (7 - 3x^2)^4$, we can employ the *chain rule* together with the *power rule* to differentiate *p* as demonstrated below.

$$p(x) = (7 - 3x^2)^4$$
$$p'(x) = 4(7 - 3x^2)^3 \cdot \frac{d}{dx}\left[7 - 3x^2\right]$$
$$p'(x) = 4(7 - 3x^2)^3 \cdot -6x$$
$$p'(x) = -24x(7 - 3x^2)^3$$

Let's consider the function $f(x) = (5x^2 - 3)(7x^3 + 2x^2 - 5)^4$. Note that *f* is the product of a quadratic function and a composition function. Accordingly, we begin to find f' by trotting out the product rule.

$$f'(x) = (5x^2 - 3) \cdot \frac{d}{dx}\left[(7x^3 + 2x^2 - 5)^4\right] + (7x^3 + 2x^2 - 5)^4 \cdot \frac{d}{dx}\left[(5x^2 - 3)\right]$$

Employing the chain rule yields f', which we write as nicely as we can after some factoring.

$$f'(x) = (5x^2 - 3) \cdot 4(7x^3 + 2x^2 - 5)^3 \frac{d}{dx}\left[7x^3 + 2x^2 - 5\right] + (7x^3 + 2x^2 - 5)^4 \cdot 10x$$
$$f'(x) = (5x^2 - 3) \cdot 4(7x^3 + 2x^2 - 5)^3 (21x^2 + 4x) + (7x^3 + 2x^2 - 5)^4 \cdot 10x$$
$$f'(x) = 2x(7x^3 + 2x^2 - 5)^3 \left[2(5x^2 - 3)(21x + 4) + 5(7x^3 + 2x^2 - 5)\right]$$
$$f'(x) = 2x(245x^3 + 50x^2 - 126x - 49)(7x^3 + 2x^2 - 5)^3$$

Lecture 9

Practice Problems

1st ed. problem set: Section 3.5 #1, #7, #15, #19, #21, #27, #31
2nd ed. problem set: Section 3.5 #3, #7, #9, #17, #21, #23
3rd ed. problem set: Section 3.5 #3, #7, #9, #17, #21, #25, #35

Possible Exam Problems

#1 Given $f(x)=\sqrt{2x^3-3}$, find $f'(x)$.

Answer: $f'(x)=\dfrac{3x^2}{\sqrt{2x^3-3}}$

#2 Given $g(x)=x^3\left(\sqrt{x}+1\right)^4$, find $g'(x)$ using the chain rule.

Answer: $f'(x)=4x^3\left(\sqrt{x}+1\right)^3\cdot\dfrac{1}{2\sqrt{x}}+\left(\sqrt{x}+1\right)^4\cdot 3x^2 = x^2\left(5\sqrt{x}+3\right)\left(\sqrt{x}+1\right)^3$

#3 Given $C(x)=\left(ax^2+bx+c\right)^n$, find $C'(x)$.

Answer: $C'(x)=n(2ax+b)\left(ax^2+bx+c\right)^{n-1}$

#4 Given $y=f(g(x))$ where $f(x)=x^2$ and $y'=6x^5+6x^2$, find $g(x)$.

Answer: $g(x)=x^3+1$

Example Exercise

Suppose $f(x) = 3x^4 \cdot (x^3 + 2x)^6$. Find $f'(x)$.

Apply the product rule.

$$f(x) = 3x^4 \cdot (x^3 + 2x)^6$$
$$f'(x) = 3x^4 \cdot \frac{d}{dx}\left[(x^3 + 2x)^6\right] + (x^3 + 2x)^6 \cdot \frac{d}{dx}\left[3x^4\right]$$

Apply the chain rule and power rule.

$$f'(x) = 3x^4 \cdot \frac{d}{dx}\left[(x^3 + 2x)^6\right] + (x^3 + 2x)^6 \cdot \frac{d}{dx}\left[3x^4\right]$$
$$f'(x) = 3x^4 \cdot 6(x^3 + 2x)^5 \frac{d}{dx}\left[x^3 + 2x\right] + (x^3 + 2x)^6 \cdot 12x^3$$
$$f'(x) = 18x^4 (x^3 + 2x)^5 (3x^2 + 2) + 12x^3 (x^3 + 2x)^6$$

Factor the greatest common factor.

$$f'(x) = 18x^4 (x^3 + 2x)^5 (3x^2 + 2) + 12x^3 (x^3 + 2x)^6$$
$$f'(x) = 6x^3 (x^3 + 2x)^5 \left[3x(3x^2 + 2) + 2(x^3 + 2x)\right]$$

Simplify.

$$f'(x) = 6x^3 (x^3 + 2x)^5 \left[3x(3x^2 + 2) + 2(x^3 + 2x)\right]$$
$$f'(x) = 6x^3 (x^3 + 2x)^5 \left[9x^3 + 6x + 2x^3 + 4x\right]$$
$$f'(x) = 6x^3 (x^3 + 2x)^5 \left[11x^3 + 10x\right]$$
$$f'(x) = 6x^3 (11x^3 + 10x)(x^3 + 2x)^5$$

Application Exercise

The chain rule states,

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right).$$

Sometimes mathematicians rewrite the chain rule using an *intermediate* variable. For a composition function like $f\left(g\left(x\right)\right)$, we introduce the intermediate variable *u*, and let $u=g\left(x\right)$. Then, we want to find the derivative of $f\left(u\right)$ where $u=g\left(x\right)$. Now, the chain rule states,

$$\frac{df}{dx}=\frac{df}{du}\cdot\frac{du}{dx}.$$

For example, suppose that a spherical balloon is being inflated or deflated so that its volume $V=\frac{4}{3}\pi r^3$ and its radius *r* are changing with time *t*. The derivative of the volume with respect to time is given as $\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$.

Accordingly, consider a balloon with volume $V=\frac{4}{3}\pi r^3$ whose radius *r* of the balloon is increasing at the rate of 0.4 cm/s when *r* = 5 cm. At what rate is the volume *V* of the balloon increasing at that instant?