

Differentiation: Product Rule and Quotient Rule

We may recall from Lecture 2 that the limit of a product is the product of limits, e.g., $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} [f(x)] \cdot \lim_{x \rightarrow a} [g(x)]$. Accordingly, we might surmise that the derivative of a product is the product of derivatives, but that is not the case.

Applying the definition of the derivative to the function $p(x) = f(x) \cdot g(x)$ we can write the following.

$$p'(x) = \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h}$$

$$p'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Adding zero in the form of $-f(x+h)g(x) + f(x+h)g(x)$ to the numerator allows us to factor as below.

$$p'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) - f(x)g(x) + f(x+h)g(x)}{h}$$

$$p'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[-f(x) + f(x+h)]}{h}$$

$$p'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$$p'(x) = \lim_{h \rightarrow 0} \left\{ f(x+h) \frac{[g(x+h) - g(x)]}{h} + g(x) \frac{[f(x+h) - f(x)]}{h} \right\}$$

Limits are linear operators, so the limit of a sum is the sum of the limits.

$$p'(x) = \lim_{h \rightarrow 0} \left\{ f(x+h) \frac{[g(x+h) - g(x)]}{h} \right\} + \lim_{h \rightarrow 0} \left\{ g(x) \frac{[f(x+h) - f(x)]}{h} \right\}$$

Unlike differentiation, as we are showing, the limit of a product does equal the product of limits.

$$p'(x) = \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Applying direct substitution, we have the following.

$$p'(x) = f(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Lecture 8

Lastly, we recognize the derivatives of f and g to arrive at the general *product rule* for differentiation.

$$p'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Product Rule: If f and g are both differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}f(x).$$

With the product rule, we can attain a rule for differentiating quotients. Assume that f and g are differentiable functions and that the quotient function $Q = f/g$ is differentiable. We can solve for f and apply the product rule as below.

$$Q(x) = \frac{f(x)}{g(x)}$$

$$Q(x) \cdot g(x) = \frac{f(x)}{\cancel{g(x)}} \cdot \cancel{g(x)}$$

$$f(x) = Q(x) \cdot g(x)$$

$$f'(x) = Q(x) \cdot g'(x) + g(x) \cdot Q'(x)$$

Solving for $Q'(x)$ and substituting for $Q(x)$, we arrive at the *quotient rule* for differentiation.

$$f'(x) = Q(x) \cdot g'(x) + g(x) \cdot Q'(x)$$

$$g(x) \cdot Q'(x) = f'(x) - Q(x) \cdot g'(x)$$

$$Q'(x) = \frac{1}{g(x)} \cdot \left[f'(x) - \frac{f(x)}{g(x)} \cdot g'(x) \right]$$

$$Q'(x) = \frac{1}{g(x)} \cdot \left[\frac{g(x) \cdot f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{g(x)} \right]$$

$$Q'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Quotient Rule: If f and g are both differentiable functions, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Practice Problems

- 1st ed. problem set: Section 3.2 #7–11 odd, #15, #17, #19, #25
 2nd ed. problem set: Section 3.2 #7–11 odd, #15, #17, #21, #27
 3rd ed. problem set: Section 3.2 #7–11 odd, #15, #17, #19, #23, #31

Possible Exam Problems

#1 Which of the following statements is true?

Statement I: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$

Statement II: $\frac{d}{dx}[f(x)/g(x)] = \frac{d}{dx} f(x) / \frac{d}{dx} g(x)$

Statement III: $\frac{d}{dx}[f(x) \cdot g(x)] = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} [g(x)]$

Statement IV: $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f(x) \cdot \frac{d}{dx} [g(x)] - g(x) \cdot \frac{d}{dx} [f(x)]}{[g(x)]^2}$

Answer: Statement III is true.

#2 Given $f(x) = x^3(\sqrt{x} + 1)$, find $f'(x)$ using the product rule.

Answer: $f'(x) = x^3 \cdot \left(\frac{1}{2} x^{-1/2} \right) + (x^{1/2} + 1)(3x^2) = \frac{7}{2} x^{5/2} + 3x^2$

#3 If $Q(x) = \frac{x^2 + 1}{x^2 - 1}$, find $Q'(x)$.

Answer: $Q'(x) = -\frac{4x}{(x^2 - 1)^2}$

Example Exercise

Suppose $f(x) = \frac{x^2 + 1}{3x + 1}$. Find $f'(x)$.

Apply the quotient rule.

$$f(x) = \frac{x^2 + 1}{3x + 1}$$

$$f'(x) = \frac{(3x + 1) \cdot \frac{d}{dx}(x^2 + 1) - [(x^2 + 1) \cdot \frac{d}{dx}(3x + 1)]}{(3x + 1)^2}$$

$$f'(x) = \frac{(3x + 1) \cdot 2x - [(x^2 + 1) \cdot 3]}{(3x + 1)^2}$$

$$f'(x) = \frac{6x^2 + 2x - [3x^2 + 3]}{(3x + 1)^2}$$

$$f'(x) = \frac{6x^2 + 2x - 3x^2 - 3}{(3x + 1)^2}$$

$$f'(x) = \frac{3x^2 + 2x - 3}{(3x + 1)^2}$$

Application Exercise

Newton's law of gravitation states that every point of mass attracts every other point mass by a force. The force is proportional to the product of the two masses, m_1 and m_2 , and inversely proportional to the square of the distance between the point masses, r . In symbols, the law of gravitation states,

$$F = G \frac{m_1 \cdot m_2}{r^2}$$

where F is the magnitude of the gravitational force between the two point masses and G is the gravitational constant. Use the quotient rule to find dF/dr for two points of mass with constant mass.