Differentiation

Calculus is a branch of mathematics derived from two problems, the first problem involved finding the instantaneous rate of change, and, as we have seen, the derivative of *f* is itself a function that yields the instantaneous rate of change of *f* for given *x*-values. The process for finding the derivative is called differentiation, and until now, we have used the definition of the derivative to perform differentiation. In this lecture, we will use d/dx notation to signify differentiation as an operation and employ some basic rules in order to find the derivative.

Differentiation as an operation has the very useful property of being a *linear operator* as detailed below.

If *f* and *g* are both differentiable functions and *c* is any real number constant, then

$$\frac{d}{dx}\left[c\cdot f(x)\right]=c\cdot\frac{d}{dx}f(x)$$

and

$$\frac{d}{dx}\left[f(x)\pm g(x)\right]=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x).$$

The most basic rule of differentiation is the *power rule* stated below.

Power Rule: If *n* is any real number, then

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}.$$

Applying the property of linearity in conjunction with the power rule, we can take the derivative of any polynomial function. Let's find the derivative of the function $p(x)=9x^3-3x+7$.

$$p'(x)=\frac{d}{dx}p(x)$$

$$p'(x)=\frac{d}{dx}\left(9x^3-3x+7\right)$$

$$p'(x)=\frac{d}{dx}\left(9x^3\right)-\frac{d}{dx}(3x)+\frac{d}{dx}(7)$$ Apply a property of linearity.

$$p'(x)=9\cdot\frac{d}{dx}\left(x^3\right)-3\cdot\frac{d}{dx}\left(x^1\right)+7\cdot\frac{d}{dx}\left(x^0\right)$$ Apply a property of linearity.

$$p'(x)=9\cdot 3x^{3-1}-3\cdot 1x^{1-1}+7\cdot 0x^{0-1}$$ Apply the power rule.

$$p'(x)=27x^2-3x^0+0$$

$$p'(x)=27x^2-3\cdot 1$$

$$p'(x)=27x^2-3$$

Notice that differentiating the linear term $-3x$ yielded -3. Notice also that the constant term 7 yielded 0.[*]

Practice Problems

1st ed. problem set: Section 3.1 #3–17 odd, #21–25 odd, #29–31 odd, #45, #49
2nd ed. problem set: Section 3.1 #3–7 odd, #11–19 odd, #31–33 odd, #47, #51
3rd ed. problem set: Section 3.1 #3–9 odd, #13–21 odd, #27, #47, #51

Possible Exam Problems

#1 Given $d(t) = \dfrac{15t^3 - 8t + 7}{t}$, find $d'(t)$.

Answer:
$$d(t) = 15t^2 - 8 + 7t^{-1}$$
$$d'(t) = 30t - 7t^{-2}$$
$$d'(t) = 30t - \frac{7}{t^2}$$
$$d'(t) = \frac{30t^3 - 7}{t^2}$$

#2 If $S(x) = f(x) + g(x)$, apply the differentiation property of linearity to find $S'(x)$.

Answer:
$$S'(x) = \frac{d}{dx} S(x)$$
$$S'(x) = \frac{d}{dx}\left[f(x) + g(x)\right]$$
$$S'(x) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$
$$S'(x) = f'(x) + g'(x)$$

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[*] Some rules for differentiation may seem quite obvious. Consider a function of the form $f(x) = c$ where *c* is any constant. The graph of *f* is a horizontal line. Any line tangent to *f* must itself be horizontal to be headed in the same direction as *f*. Consequently, the derivative of any constant function must be a constant function equal to zero. In other words, if $f(x) = c$, then $f'(x) = 0$. Similarly, consider a linear function of the form $g(x) = mx + b$ where *m* is a non-zero number that indicates the slope of the line. To head in the same direction as *g*, any line tangent to *g* must coincide with *g* and share its slope; thus, *g*' is a constant function equal to *m*, the slope of *g*. Accordingly, if $g(x) = mx + b$, then $g'(x) = m$.

Example Exercise

Suppose $f\left(x\right)=\dfrac{x^4+8x}{x}$. Find the equations of all lines tangent to $f\left(x\right)$ with a slope equal to 12.

Find $\dfrac{d}{dx}\left[\dfrac{x^4+8x}{x}\right]$ as below.

$$\frac{d}{dx}\left[\frac{x^4+8x}{x}\right]=\frac{d}{dx}\left[\frac{x^4}{x}+\frac{8x}{x}\right]=\frac{d}{dx}\left[x^3+8\right]=\frac{d}{dx}\left[x^3\right]+\frac{d}{dx}[8]=3x^2+0=3x^2$$

Set $\dfrac{d}{dx}\left[\dfrac{x^4+8x}{x}\right]$ equal to 12 and solve.

$$\begin{aligned}3x^2&=12\\x^2&=4\\x&=\pm 2\end{aligned}$$

When $x=\pm 2$, the tangent line has a slope of twelve. Find the two points of tangency.

$$f\left(2\right)=\frac{\left(2\right)^4+8\left(2\right)}{\left(2\right)}=\frac{16+16}{2}=16\Rightarrow\left(2,16\right)$$

$$f\left(-2\right)=\frac{\left(-2\right)^4+8\left(-2\right)}{\left(-2\right)}=\frac{16-16}{-2}=0\Rightarrow\left(-2,0\right)$$

Substitute into the point-slope formula for a linear equation.

$$\begin{aligned}y-y_1&=m\left(x-x_1\right)\\y-16&=12\left(x-2\right)\\y-16&=12x-24\\y&=12x-8\end{aligned}$$

$$\begin{aligned}y-y_1&=m\left(x-x_1\right)\\y-0&=12\left(x-\left(-2\right)\right)\\y&=12\left(x+2\right)\\y&=12x+24\end{aligned}$$

Application Exercise 1

In physics, linear density equals the rate of change of mass with respect to length. If *m* represents mass, *x* represents length, and ρ represents linear density, then $\rho = dm/dx$.

Suppose $m = \sqrt{x}$ where *m* represents the mass of an object in kilograms and *x* represents length measured in meters. Calculate ρ when the length of the object equals one meter.

Application Exercise 2

In 1840, French physician Jean Louis Marie Poiseuille and German physicist Gotthilf Heinrich Ludwig Hagen formulated the law of laminar flow, which gives the velocity of fluids moving through a tube. Let *P* represent the pressure difference between the ends of a blood vessel of length *l* and radius *R*. Let η represent the viscosity of the blood in the vessel. Then, Poiseuille's law states,

$$v = \frac{P}{4\eta l}\left(R^2 - r^2\right) .$$

When pressure, length, viscosity, and radius are all constant, the velocity of the blood is a function of *r*, the blood's distance from the central axis of the blood vessel.

The velocity gradient equals dv/dr. For a particular human artery, $\eta = 0.027$, $R = 0.008 \text{ cm}$, $l = 2 \text{ cm}$, and $P = 4000 \text{ dynes/cm}^2$. Find the velocity gradient for this particular artery when $r = 0.002 \text{ cm}$.