

### Derivative as a Function

Lectures 4 and 5 approached the derivative from two points of view. First, we imagined the derivative as the "steepness" of the curve, the slope of the tangent line. Second, we imagined the derivative as the instantaneous rate of change. In both cases, we started by defining the derivative as a point property. The derivative gives the steepness of the curve at some point  $a$ . The derivative gives the instantaneous rate of change of the function at some point  $a$ . Also, in both cases, we noted that finding the derivative in general leads to thinking of the derivative as a function that gives the steepness of the curve or the instantaneous rate of change of the function for any given value of  $x$ . In this section, we will explicitly think of the derivative as a function according to the definition below.

The derivative function of  $f(x)$ , denoted  $f'(x)$ , is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of the derivative is some subset of the domain of  $f$  and is restricted to  $x$ -values where the limit above exists and where  $f$  is defined.

To apply the definition, remember that  $h$  is the variable for taking the limit and  $x$  is temporarily some constant. Consider the function  $g(x) = \sqrt{1-x}$  defined on the interval  $(-\infty, 1]$ . Let's apply the definition above to find  $g'$ .

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ g'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h} \\ g'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{1-x-h} - \sqrt{1-x}}{h} \cdot \frac{\sqrt{1-x-h} + \sqrt{1-x}}{\sqrt{1-x-h} + \sqrt{1-x}} \\ g'(x) &= \lim_{h \rightarrow 0} \frac{1-x-h - (1-x)}{h(\sqrt{1-x-h} + \sqrt{1-x})} \\ g'(x) &= \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{1-x-h} + \sqrt{1-x})} \\ g'(x) &= \frac{-1}{(\sqrt{1-x-0} + \sqrt{1-x})} \\ g'(x) &= \frac{-1}{2\sqrt{1-x}} \end{aligned}$$

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Our result,  $g'(x) = \frac{-1}{2\sqrt{1-x}}$ , is a function defined on the interval  $(-\infty, 1)$ . Notice the restriction at  $x = 1$ . The domain of  $g'$  is a proper subset of the domain of  $g$ . Recall that the derivative is a point property defined as

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} \text{ where the limit exists.}$$

If we let  $a = 1$ , we run into some difficulty.

$$\begin{aligned} g'(1) &= \lim_{h \rightarrow 0} \frac{g(1+h) - g(1)}{h} \\ g'(1) &= \lim_{h \rightarrow 0} \frac{\sqrt{1-(1+h)} - \sqrt{1-(1)}}{h} \\ g'(1) &= \lim_{h \rightarrow 0} \frac{\sqrt{-h}}{h} \end{aligned}$$

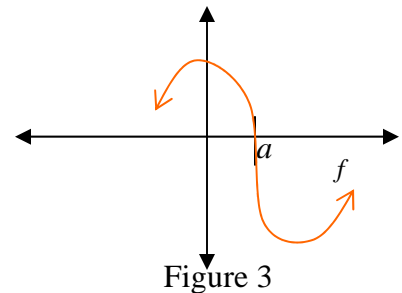
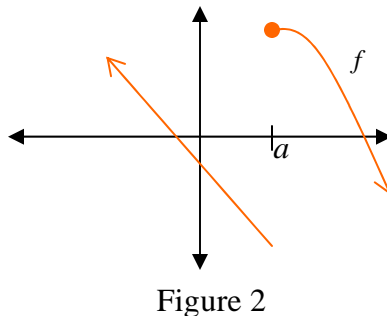
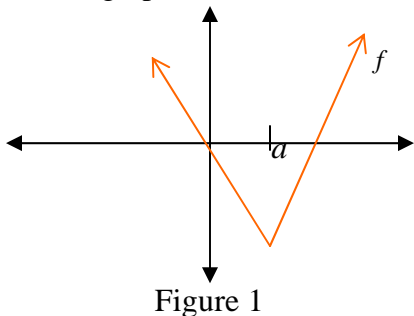
Here the limit becomes problematic. Direct substitution yields  $\frac{0}{0}$ , which is undefined. The difficulty stems from the fact that the limit simply does not exist. In such a case, the function is said to be non-differentiable at  $x = a$ . This result leads to the definition below.

A function  $f$  is differentiable at  $a$  if  $f'(a)$  exists. It is differentiable on an open interval  $(a, b)$  or  $(a, \infty)$  or  $(-\infty, a)$  or  $(-\infty, \infty)$  if it is differentiable at every number in the interval.

Careful readers may have noticed that while  $g$  is *continuous from the left* at  $x = 1$ , it is not *continuous* at  $x = 1$ . Is continuity related to differentiability? It turns out, yes, and the following theorem makes the relation explicit.

**Differentiability Theorem:** If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

Note, however, that the converse is not true. If a function is continuous at  $a$ , it is not necessarily differentiable at  $a$ . The absolute value function  $f(x) = |x|$  provides an example of this. The absolute value function is continuous for all values of  $x$ , but it is not differentiable at  $x = 0$ . The three graphs below reveal functions that are non-differentiable at  $x = a$ .



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A function is non-differentiable at any point where the function has a "kink" or "corner" as illustrated in Figure 1 at  $x = a$ . A function is non-differentiable at any point where the function is not defined as illustrated in Figure 2 at  $x = a$ . A function is non-differentiable at any point where the tangent line is vertical as illustrated in Figure 3 at  $x = a$ .

We have established that the derivative of  $f$ , denoted  $f'$ , is itself a function. Its inputs include all the  $x$ -values in the domain of  $f$  where  $f$  is differentiable. Its outputs are both the slopes of the lines tangent to  $f$  at a given input and the instantaneous rate of change of  $f$  at any given input. Intuitively then the reader may suspect that derivatives may themselves be differentiable. If so, what do we call the derivative of a derivative? We call them higher order derivative. Consider the example of a position function for a falling object such as  $s(t) = 4.9t^2$ . The derivative of  $s$  is the velocity function  $v(t) = 9.8t$ . The derivative of  $v$  is the acceleration function  $a(t) = 9.8$  (falling objects accelerate at a constant rate). The acceleration function is the second derivative of the position function. We denote the second derivative of  $s$  as  $s''$  such that  $a(t) = v'(t) = s''(t)$ .

Before finding higher order derivatives, we must turn to the issue of notation. If we let  $y = f(x)$ , the derivative of  $f(x)$  can be written as

$$f'(x) \text{ or } \frac{df}{dx} \text{ or } \dot{f} \text{ or } D_x f(x) \text{ or } y' \text{ or } \frac{dy}{dx} \text{ or } Df(x) \text{ or } \frac{d}{dx} f(x).$$

The notation  $df/dx$  is due to Leibnitz and is suggestive of a ratio of a change in  $y$  to a change in  $x$ :

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

The symbols  $D$  and  $d/dx$  are called differentiation operators because they indicate the operation of differentiation.

With higher order derivatives, the various notations can still be employed. For example, we write the second derivative of  $y = f(x)$  as

$$f''(x) \text{ or } f^2(x) \text{ or } \frac{d}{dx} \left( \frac{dy}{dx} \right) \text{ or } \frac{d^2 y}{dx^2}.$$

Consider the function  $f(x) = x^3 + x$  whose derivative is  $f'(x) = 3x^2 + 1$ . To find the third derivative of  $f$ , we must start by finding the second derivative as shown below.

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$$\begin{aligned}f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\f''(x) &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 + 1 - (3x^2 + 1)}{h} \\f''(x) &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 1 - 3x^2 - 1}{h} \\f''(x) &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} \\f''(x) &= \lim_{h \rightarrow 0} 6x + 3h \\f''(x) &= 6x + 3(0) \\f''(x) &= 6x\end{aligned}$$

The derivative of the second derivative is the third derivative.

$$\begin{aligned}f'''(x) &= \lim_{h \rightarrow 0} \frac{f''(x+h) - f''(x)}{h} \\f'''(x) &= \lim_{h \rightarrow 0} \frac{6(x+h) - 6x}{h} \\f'''(x) &= \lim_{h \rightarrow 0} \frac{6x + 6h - 6x}{h} \\f'''(x) &= \lim_{h \rightarrow 0} \frac{6h}{h} \\f'''(x) &= \lim_{h \rightarrow 0} 6 \\f'''(x) &= 6\end{aligned}$$

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### Practice Problems

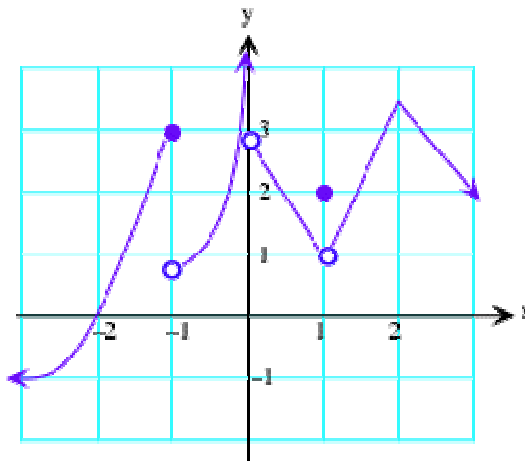
- 1st ed. problem set: Section 2.8 #19–23 odd & #29  
2nd ed. problem set: Section 2.8 #19–25 odd & #31  
3rd ed. problem set: Section 2.8 #19–25 odd & #31, #33

### Possible Exam Problems

#1 Given the position function  $s(t) = 128t - 16t^2$ , find and interpret  $s''(t)$ .

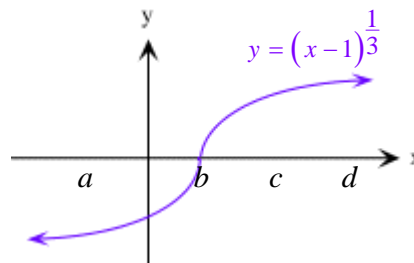
Answer:  $s''(t) = -32$  and represents the acceleration function of the object whose position is given by  $s$ .

#2 Name the points where the function graphed below is non-differentiable.



Answer:  $-1, 0, 1, 2$

#3 Given the graph of function  $y$  below, name the  $x$ -values for which  $dy/dx$  does not exist.



Answer: The function is non-differentiable at  $b$  because the tangent line will be vertical.

### Example Exercise 1

Use the Differentiability Theorem to prove  $f(x) = 2x^2 + 3$  is continuous at  $x = 2$ .

Consider  $f'(2)$ . By definition,

$$\begin{aligned}
 f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 f'(2) &= \lim_{h \rightarrow 0} \frac{2(2+h)^2 + 3 - [2(2)^2 + 3]}{h} \\
 f'(2) &= \lim_{h \rightarrow 0} \frac{2(4 + 4h + h^2) + 3 - [11]}{h} \\
 f'(2) &= \lim_{h \rightarrow 0} \frac{8 + 8h + 2h^2 - 8}{h} \\
 f'(2) &= \lim_{h \rightarrow 0} \frac{8h + 2h^2}{h} \\
 f'(2) &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8 + 2h)}{\cancel{h}} \\
 f'(2) &= \lim_{h \rightarrow 0} (8 + 2h) \\
 f'(2) &= 8
 \end{aligned}$$

We see that  $f'(2)$  is defined. By Differentiability Theorem, if  $f'$  is defined at  $a$ , then  $f$  is continuous at  $a$ ; therefore,  $f$  is continuous at 2 since  $f'(2)$  is defined.

## Example Exercise 2

Suppose  $f(x) = |x+2|$ . Find the domain of  $f'(x)$ .

Recall the definition for absolute value.

$$f(x) = |x+2| = \sqrt{(x+2)^2}$$

Find the derivative of  $f$ .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h+2)^2} - \sqrt{(x+2)^2}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h+2)^2} - \sqrt{(x+2)^2}}{h} \cdot \frac{\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2}}{\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2}}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h+2)^2 - (x+2)^2}{h(\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + 4x + h^2 + 4h + 4 - (x^2 + 4x + 4)}{h(\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2 + 4h}{h(\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2})}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 4)}{\cancel{h}(\sqrt{(x+h+2)^2} + \sqrt{(x+2)^2})}$$

$$f'(x) = \frac{2x + 4}{\sqrt{(x+2)^2} + \sqrt{(x+2)^2}}$$

$$f'(x) = \frac{\cancel{2}(x+2)}{\cancel{2}\sqrt{(x+2)^2}}$$

$$f'(x) = \frac{(x+2)}{\sqrt{(x+2)^2}} = \frac{x+2}{|x+2|}$$

Note that the derivative is undefined at  $x = -2$ . Hence, the domain in set interval notation is  $(-\infty, -2) \cup (-2, \infty)$ .

### Application Exercise

The *jerk* is the third derivative of a position function. Thus, the jerk is the derivative of an acceleration function. A large jerk indicates a sudden change in acceleration corresponding to abrupt movement in the subject of the position function.

The function  $v(t) = 0.0013t^3 - 0.0903t^2 + 23.61t - 3.083$  gives the velocity in feet per second of a space shuttle on a particular mission from lift off at time  $t = 0$  to time  $t = 126$  seconds after lift off. At what moment does the shuttle experience a jerk of  $0.5 \text{ ft/s}^3$  ?