

Derivative as Instantaneous Rate of Change

Consider a function that describes the position of a racecar moving in a straight line away from some starting point. Let $y = s(t)$ such that t represents the time in hours since the race began and $s(t)$ represents the distance in miles the car has driven from its starting position. In this scenario, the point $(0,0)$ indicates that the car is zero miles from the starting point after zero hours, and the point $(1,240)$ indicates that the car is 240 miles from the starting point after one hour. Such a function is a position function.

Quite naturally, the driver of the racecar is interested in his/her speed, which is the rate of change of the car's position. Indeed, car speeds in miles per hour equal an average, namely, the distance traveled divided by the elapsed interval of time. If the racecar is 240 miles from the starting point after one hour of driving, then the car's average rate of speed must have been 240 miles per hour. In general, consider the time interval from $t = a$ to $t = a + h$. Over this interval, the change in position is $s(a+h) - s(a)$, and the average velocity is given by

$$\frac{\text{change in position}}{\text{time elapsed}} = \frac{s(a+h) - s(a)}{(a+h) - a} = \frac{s(a+h) - s(a)}{h}.$$

At any given point during the race, however, the driver is less interested in his average speed and more interested in his speed at that moment, his/her instantaneous velocity. We can approach instantaneous velocity by computing the average velocities over shorter and shorter intervals of time—that is, as h approaches zero; thereby, arriving at the instantaneous velocity by taking the limit as defined below.

Given a position function $y = s(t)$, the instantaneous velocity at point a , denoted $v(a)$, is given by

$$v(a) = \lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h}.$$

Comparing the definition above to the definition of the slope of a tangent line, we see that instantaneous velocity at time a (the instantaneous rate of change) is equivalent to the slope of a line tangent to the position function at time a , which we defined as the derivative. If a is some point in general, then $v(a)$ is actually a function. Replacing a with the variable t , we have the definition for a velocity function, the derivative of a position function.

Given a position function $y = s(t)$, the instantaneous velocity of the moving object is a function of time given by the derivative of s .

$$v(t) = s'(t) = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}.$$

In the preceding discussion, we found the instantaneous velocity, which was really the instantaneous rate at which the position changed. In general then, for any given function $y = f(x)$, the instantaneous rate of change of the y -values equals the limit of the ratio of change

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in y to change in x as the change in x approaches zero. In symbols, we write $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x$.

Allowing h to represent the change in x , we have:

$$\text{instantaneous rate of change} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Note that the instantaneous rate of change equals the derivative of f , so we have another interpretation of the derivative. Geometrically, the derivative is a function f' that gives the slope of a line tangent to the function f at any given point. Also, the derivative is a function f' that gives the instantaneous rate of change of the function f at any given point. If f is a position function, f' is the velocity function.

Just as the velocity function is the derivative of the position function, the acceleration function is the derivative of the velocity function. In general, if $y = s(t)$ gives the position of an object, $v(t)$ gives the velocity of the object, and $a(t)$ gives the acceleration of the object, then $v(t) = s'(t)$ and $a(t) = v'(t)$.

Practice Problems

1st ed. problem set: Section 2.6 #13–17 odd, Section 2.7 #23

2nd ed. problem set: Section 2.6 #13–17 odd, Section 2.7 #25

3rd ed. problem set: Section 2.6 #15–19 odd, Section 2.7 #25

Possible Exam Problem

Given the position function $s(t) = 4.9t^2$, use the definition of instantaneous velocity to find the velocity function.

Answer: $v(t) = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$

$$v(t) = \lim_{h \rightarrow 0} \frac{4.9(t+h)^2 - 4.9t^2}{h}$$

$$v(t) = \lim_{h \rightarrow 0} \frac{4.9(t^2 + 2th + h^2) - 4.9t^2}{h}$$

$$v(t) = \lim_{h \rightarrow 0} \frac{9.8th + 4.9h^2}{h}$$

$$v(t) = \lim_{h \rightarrow 0} 9.8t + 4.9h$$

$$v(t) = 9.8t$$

Example Exercise

$S(t) = 5t^2 + 3t - 10$ describes the position of an object in meters after t seconds. Find the velocity of the object after three seconds.

The velocity equals the derivative of the position function.

$$S(t) = 5t^2 + 3t - 10$$

$$v(3) = S'(3) = \lim_{h \rightarrow 0} \frac{S(3+h) - S(3)}{h}$$

$$v(3) = \lim_{h \rightarrow 0} \frac{5(3+h)^2 + 3(3+h) - 10 - [5(3)^2 + 3(3) - 10]}{h}$$

$$v(3) = \lim_{h \rightarrow 0} \frac{5(9 + 6h + h^2) + 9 + 3h - 10 - [44]}{h}$$

$$v(3) = \lim_{h \rightarrow 0} \frac{45 + 30h + 5h^2 + 3h - 45}{h}$$

$$v(3) = \lim_{h \rightarrow 0} \frac{33h + 5h^2}{h}$$

$$v(3) = \lim_{h \rightarrow 0} \frac{\cancel{h}(33 + 5h)}{\cancel{h}}$$

$$v(3) = \lim_{h \rightarrow 0} (33 + 5h)$$

$$v(3) = 33$$

The velocity equals 33 meters per second.

Application Exercise

The function $A = 58t - 0.83t^2$ gives the height of a projectile with an initial velocity of 58 meters per second. What is the velocity of the projectile after one second?