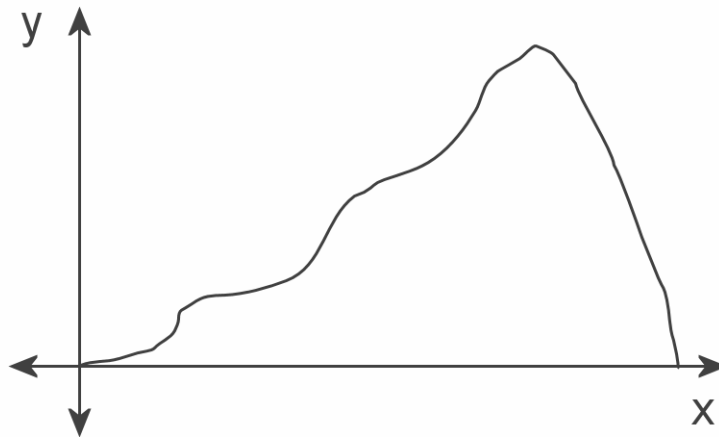


Derivative as Slope

Historically, one of the two big problems in calculus involves finding instantaneous rate of change. The concept that solves this problem is called the derivative. One way to conceive the derivative is to think of a curve's "slope" at any given point. Slope, of course, is a number assigned to a line to describe the line's steepness. The slope of a line is constant and therefore thinking of a line's slope is natural. Thinking of a curve's slope is not so natural.

Consider a function that describes the surface of a mountain. Try to imagine poor Sisyphus pushing his boulder up the mountain. Sisyphus was the clever and devious character in Greek mythology who flouted Greek traditions of hospitality by murdering his guests. The gods eventually damned Sisyphus for his audacity. Sisyphus's punishment requires that he eternally push a giant boulder up a mountain to the peak, only to have the boulder roll down so that he must begin his toil anew.



Suppose that Sisyphus starts his task at the base of the mountain with coordinates $(0,0)$. As he moves up the mountain, both x and y coordinates will increase. The function $y = h(x)$ gives the height of the mountain above point x , so the graph of $h(x)$ is the figure shown above.

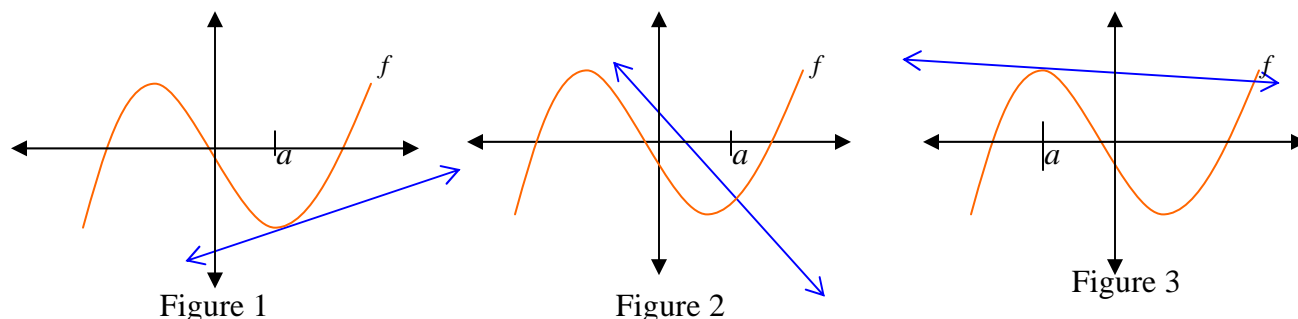
Pitiable Sisyphus is captivated by the steepness of the mountain at any given point. The derivative of $h(x)$, denoted $h'(x)$ or $\frac{dh}{dx}$, equals the steepness at point x . For instance, suppose $h'(a) = c/b$. After having reached a point that is a units in the x -direction from the base of the mountain, Sisyphus is at a point on the mountain where the steepness is c/b . At this point, Sisyphus must rise c units in order to move b units in the x -direction from $(0,0)$. Not all points on $h(x)$ have the same steepness, however. Suppose $h'(d) = b/c$. Now, Sisyphus must rise b units in order to move c units in the x -direction from $(0,0)$. If $b = 10$ and $c = 1$, Sisyphus strains much less at $h(a)$ than he does at $h(d)$.

Of course, there is an instance where Sisyphus has no strain at all. This instance occurs at the peak where the mountain has no steepness whatsoever. If the peak occurs p units in the x -direction from $(0,0)$, then $h'(p) = 0$. At this point, Sisyphus has no further climbing, and Camus imagines that he gets a little giddy. Part of Sisyphus's doom, however, requires that he must descend the mountain to catch up to the boulder, which has tumbled down the far-side of the

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mountain. For points that are further than p units in the x -direction from $(0,0)$, the steepness is negative, indicating that Sisyphus must descend some number of units in order to move in the x -direction. If $q > p$, then $h'(q)$ equals a negative ratio.

This idea of a curve's steepness can be captured in the idea of the slope of a tangent line. In layman's terms, a *tangent* line is a line that barely skims a curve touching at some singular point of interest and heading in the same direction as the curve. The lines in Figure 1 and Figure 3 below are tangent to the curve described by $f(x)$ at the point $x = a$. Despite the intersection at the point where $x = a$, the line in Figure 2 is not tangent to f because it does not head in the same direction as f .



The fact that the line in Figure 3 happens to touch the curve at some point other than where $x = a$ is nonmaterial.

It turns out that the derivative of f at the point $x = a$ equals the slope of the line tangent to f at $x = a$. Based on our discussion of Sisyphus above, this may seem intuitive, but it leaves us with a problematic question: how do we calculate the slope of a line with only one known point?

From elementary algebra, we recall that the slope of any line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, or, if we

let $y = f(x)$, then $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. So, calculating the slope of a line given two points is

simple enough. To find the derivative, however, we must calculate the slope of a line given only one point, which is problematic. A solution is at hand and requires only two things. First, the idea of a *secant* line. Second, our friend from previous lectures, the limit.

A secant line intersects a curve at two points of interest; therefore, calculating the slope of a secant is mundane. To find the slope of a line tangent to some function f at the point $(a, f(a))$, we consider some nearby point $(a+h, f(a+h))$ where h represents some tiny distance along the x -axis. Accordingly, the slope of the secant line is given below.

$$m = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Now, we introduce our friend, the limit, to our friend, the slope of the secant line, and say that the tangent line is the limiting position of the secant line as h approaches zero as defined below.

The slope of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

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This definition then becomes our definition for the derivative of f at $x = a$.

The derivative of function f at a number a , denoted $f'(a)$, is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists, in which case, the function is said to be differentiable at a .
If this limit does not exist, the function is said to be un-differentiable at a .

Using the definitions above, we can find the derivative of $f(x)$ at $x = a$ and/or the slope of a line tangent to $f(x)$ at $x = a$. Consider the function $f(x) = 1/x$. The slope of the line tangent to $f(x)$ at $x = a$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ f'(a) &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} \cdot \frac{a}{a} - \frac{1}{a} \cdot \frac{a+h}{a+h}}{h} \\ f'(a) &= \lim_{h \rightarrow 0} \frac{\frac{a}{a(a+h)} - \frac{a+h}{a(a+h)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{a - (a+h)}{a(a+h)}}{h} \\ f'(a) &= \lim_{h \rightarrow 0} \frac{\frac{-h}{a(a+h)}}{h} = \lim_{h \rightarrow 0} \frac{-h}{a(a+h)} \div h \\ f'(a) &= \lim_{h \rightarrow 0} \frac{\cancel{-h}}{a(a+h)} \cdot \frac{1}{\cancel{h}} = \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} \\ f'(a) &= \frac{-1}{a(a+0)} = -\frac{1}{a^2} \end{aligned}$$

Notice that the above calculation is valid for any choice of a in the domain of $f(x) = 1/x$. If we consider a to be any point in the domain, then we may view the derivative itself as a function. With this change of perspective in mind, we will replace a with a variable x to obtain the definition for a function called the derivative of f , whose value at any given point can be interpreted as the slope of the line tangent to f at that point.

The derivative function of $f(x)$, denoted $f'(x)$, is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

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Practice Problems in *Calculus: Concepts and Contexts* by James Stewart

- 1st ed. problem set: Section 2.6 #1, #3, #5.a.ii, #5.b, #7, #9
 Section 2.7 #3–9 odd, #13–17 odd
- 2nd & 3rd ed. problem set: Section 2.6 #1, #3, #5.a.ii, #5.b, #7, & #9
 Section 2.7 #3–9 odd, #13–19 odd

Practice Problems in *Calculus: Early Transcendentals* by Briggs and Cochran

- 1st ed. problem set: 3.1 #17-29 odd, #53

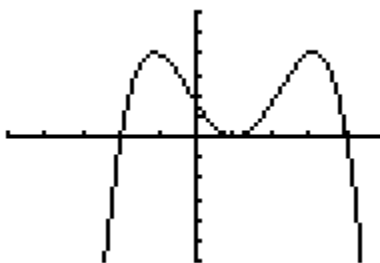
Possible Exam Problems

- #1 Given the graph of $y = f(x)$ below, describe the intervals where $f'(x) > 0$.



Answer: $(-\infty, -1) \cup (1, 3)$

- #2 Given the graph of $y = f(x)$ below, name the x -values where $f'(x) = 0$.



Answer: $x = -1, 1, 3$

- #3 Use the definition of the derivative, to find $f'(a)$ given $f(x) = 3x^2 + 2$.

$$\text{Answer: } f'(a) = \lim_{h \rightarrow 0} \frac{3(a+h)^2 + 2 - (3a^2 + 2)}{h} = 6a$$

- #4 Find the equation of the line tangent to $g(x) = 5x^2 - 1$ at $x = 4$.

Answer: $y = 40x - 81$

Example Exercise 1

Suppose $f(x) = x^2 + 2x$. Use the definition of the derivative to find $f'(1)$.

State the definition and substitute.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{(1+h)^2 + 2(1+h) - [(1)^2 + 2(1)]}{h}$$

Simplify.

$$f'(1) = \lim_{h \rightarrow 0} \frac{(1+h)^2 + 2(1+h) - [(1)^2 + 2(1)]}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 + 2 + 2h - [3]}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{h^2 + 4h}{h}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{\cancel{h}(h+4)}{\cancel{h}}$$

$$f'(1) = \lim_{h \rightarrow 0} (h+4)$$

$$f'(1) = 4$$

Example Exercise 2

Suppose $f(x) = \frac{2x}{x+3}$. Use the definition of the derivative to find $f'(2)$.

State the definition and substitute.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{\frac{2(2+h)}{(2+h)+3} - \frac{2(2)}{2+3}}{h}$$

Simplify.

$$f'(2) = \lim_{h \rightarrow 0} \frac{\frac{4+2h}{5+h} - \frac{4}{5}}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{\frac{4+2h}{5+h} \cdot \left(\frac{5}{5}\right) - \frac{4}{5} \cdot \left(\frac{5+h}{5+h}\right)}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{\frac{20+10h}{5(5+h)} - \frac{20+4h}{5(5+h)}}{h}$$

$$f'(2) = \lim_{h \rightarrow 0} \left[\frac{20+10h-20-4h}{5(5+h)} \div h \right]$$

$$f'(2) = \lim_{h \rightarrow 0} \left[\frac{6h}{5(5+h)} \div h \right]$$

$$f'(2) = \lim_{h \rightarrow 0} \left[\frac{6h}{5(5+h)} \cdot \frac{1}{h} \right]$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{6}{5(5+h)}$$

$$f'(2) = \frac{6}{25}$$

Example Exercise 3

Find the equation of the line tangent to $f(x) = x - x^2$ when $x = 1/4$.

To find the equation of a line, we need the line's slope and a point that the line passes through. We start by finding the slope, i.e., find $f'(1/4)$.

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \frac{\left(\frac{1}{4} + h\right) - \left(\frac{1}{4} + h\right)^2 - \left[\frac{1}{4} - \left(\frac{1}{4}\right)^2\right]}{h}$$

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \frac{\frac{1}{4} + h - \left(\frac{1}{16} + \frac{h}{2} + h^2\right) - \left[\frac{3}{16}\right]}{h}$$

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \frac{\frac{1}{4} + h - \frac{1}{16} - \frac{h}{2} - h^2 - \frac{3}{16}}{h}$$

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \frac{-h^2 + \frac{h}{2}}{h}$$

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \frac{\cancel{h} \left(-h + \frac{1}{2}\right)}{\cancel{h}}$$

$$f'\left(\frac{1}{4}\right) = \lim_{h \rightarrow 0} \left(-h + \frac{1}{2}\right)$$

$$f'(1/4) = 1/2$$

Find the point of tangency, $f(1/4) = 3/16$, and substitute into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

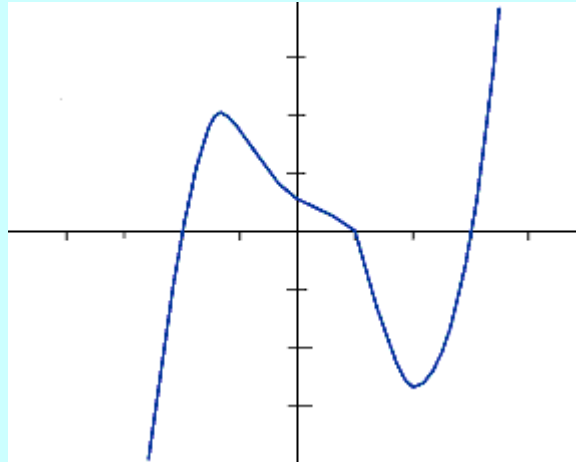
$$y - \frac{3}{16} = \frac{1}{2} \left(x - \frac{1}{4}\right)$$

$$y = \frac{1}{2}x - \frac{1}{8} + \frac{3}{16}$$

$$y = \frac{1}{2}x + \frac{1}{16}$$

Example Exercise 4

Consider the graph of $f(x)$ below.



Arrange the following values in order from least to greatest.

$$f(0), f'\left(\frac{1}{2}\right), f'(2), f(-2), f'(-2)$$

Approximate the values of the function.

$$f(0) \approx 0.5$$

$$f(-2) = 0$$

Approximate the slope of the curve by imagining tangent lines for the values of the derivative.

$$f'\left(\frac{1}{2}\right) \approx -0.5$$

$$f'(2) = 0$$

$$f'(-2) \approx 2$$

Arrange values from least to greatest.

$$f'\left(\frac{1}{2}\right), f(-2) \& f'(2), f(0), f'(-2)$$

Application Exercise

Consider the elevation in thousands of feet of a physical formation given by the function below where x represents the distance in miles due north from point P , a particular location at the base of the formation.

$$E(x) = 20x - 10x^2$$

Imagine a hiker at point P who will hike due north over the physical formation whose elevation is given by $E(x)$. How far due north with respect to the horizontal distance (i.e., along the x -axis) must the hiker travel before his hike changes from an ascent to a descent?