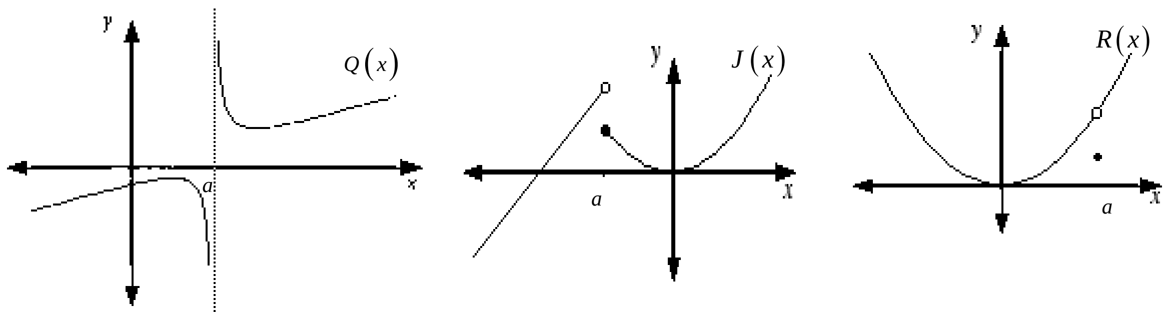
Continuity

Continuity is a function characteristic. Some functions are continuous over the entire number line, $\mathbb{R}$. Others are continuous only for some proper subset of the number line. Continuity implies an absence of interruptions. If a function is continuous on an interval its graph is unbroken on the interval. If a function is not continuous, its graph is broken and said to be discontinuous. Formally, continuity requires three conditions.

A function *f* is **continuous at a number *a*** if

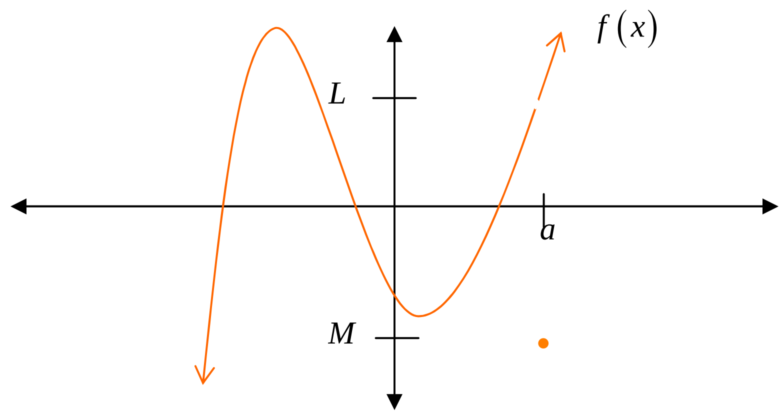
1. $f(a)$ is defined,
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$.

If a function fails condition one or condition two at some point, then the function also fails condition three at that point. The three graphs below each demonstrate a failure to satisfy at least two of the three conditions for continuity at point *a*. $Q(x)$ fails condition one, for the function is not defined at *a*. (It also fails condition two since the limit as *x* approaches *a* does not exist.) $Q(x)$ exhibits a type of discontinuity called *infinite discontinuity*. $J(x)$ satisfies condition one but fails condition two at point *a* since the limit as *x* approaches *a* does not exist. $J(x)$ exhibits *step* or *jump discontinuity* so called because the function "steps" or "jumps" from one value to another. $R(x)$ satisfies conditions one and two, but not condition three at point *a* because $\lim_{x\to a} R(x) \neq R(a)$. $R(x)$ exhibits *removable discontinuity* so called because the function could be made continuous by redefining it at a single point.



Due to the reliance on limits, the definition of continuity simultaneously stresses a single point and an interval of points. Ambivalence abounds: continuity is a point property that occurs within an interval of defined values of *f*.

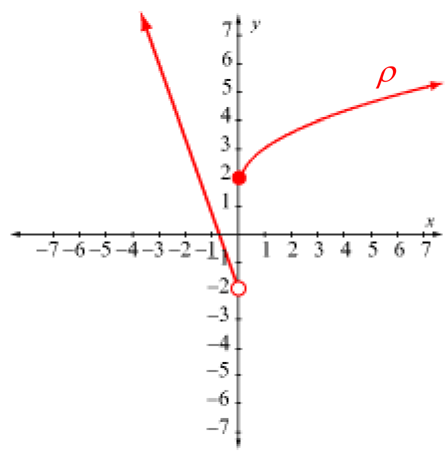
Consider the graph of *f* below. The graph exhibits a break or interruption when *x* = *a*. We say, "Function *f* is discontinuous at $x = a$." The limit of *f* as *x* approaches *a* equals *L*, but $f(a) = M$. Since $\lim_{x\to a} f(x) \neq f(a)$, the third condition for continuity is not met.



To describe the intervals of continuity, we write $(-\infty, a) \cup (a, \infty)$. Notice the use of a parenthesis with the number *a*. There must be an emphasis on the open interval because *f* is not continuous at *a*.

> A function is **continuous on the open interval (*a*,*b*)** if it is continuous at every point in the closed interval [*a*,*b*] except possibly at the endpoints *a* and *b*.

Consider the graph of ρ below. The graph exhibits a break or interruption when *x* = 0. We say, "Function ρ is discontinuous at $x = 0$." The limit of ρ as *x* approaches zero does not exist, so the function fails condition two when $x = 0$.



Despite the discontinuity at $x = 0$, we can say, "Function ρ is continuous on the open intervals $(-\infty, 0) \cup (0, \infty)$." Intuitively, there seems to be a difference between the interval $(-\infty, 0)$ versus the interval $(0, \infty)$. To draw a distinction between these two intervals, we will apply the following definitions.

A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a),$$

and f is **continuous from the left at a number *a*** if

$$\lim_{x \to a^-} f(x) = f(a).$$

Now, we can deal with functions like ρ on the interval $[0,\infty)$. We will say, "Function ρ is continuous on the interval $[0,\infty)$." To make this statement, we need the definition below.

A function f is **continuous on an interval [*a*,*b*)** if it is continuous at every number within the interval (a,b) and is either *continuous* or *continuous from the right* at a. Similarly, a function f is **continuous on an interval (*a*,*b*]** if it is continuous at every number within the interval (a,b) and is either *continuous* or *continuous from the left* at b.

To establish the continuity of a function over an interval using the definitions from above, it is necessary to establish that the function is continuous at an infinite number of x-values, a daunting task to say the least. Accordingly, we will establish the continuity of some functions by accepting a few theorems. Some functions are continuous over the entire number line.

$\mathbb{R}$ **-Continuous Theorem**: Exponential functions, sine-wave and cosine-wave functions, and polynomial functions are continuous over the interval $(-\infty,\infty)$.

Some functions that not continuous over the number line are continuous over their domain.

***D*-Continuous Theorem**: Root functions, rational functions, logarithmic functions, and the remaining trigonometric functions are all continuous over their domains.

Functions derived from operations with continuous functions are themselves continuous.

Continuity with Function Operations Theorem: If f and g are continuous at a, then the following functions are continuous at a:

1. $f \pm g$
2. fg
3. $\dfrac{f}{g}$ if $g(a) \neq 0$
4. $c \cdot f$ where c is a constant.

Using the above theorems, we can show that $y=(\sin x)/x$ is continuous for all real numbers except zero. Let $f(x)=\sin x$ and $g(x)=x$. Function *f* is the sine-wave function, which is continuous by the $\mathbb{R}$-Continuous Theorem. Function *g* is a first-degree polynomial function, which is also continuous by the $\mathbb{R}$-Continuous Theorem. By the Continuity with Function Operations Theorem, we know $\sin x/x$ is continuous for all *x* except zero.

Practice Problems

1st ed. problem set:	2.4 #1–5 odd, #11–17 odd
2nd ed. problem set:	2.4 #1–5 odd, #9, #13–19 odd
3rd ed. problem set:	2.4 #1–5 odd, #9, #13–19 odd

Possible Exam Problems

#1 What are the three conditions for continuity at point *a*.

Answer: 1. $f(a)$ is defined, 2. $\lim_{x\to a} f(x)$ exists, 3. $\lim_{x\to a} f(x)=f(a)$.

#2 Consider $L(x)=\sqrt{x+1}$. Is *L*(*x*) continuous on the following intervals?

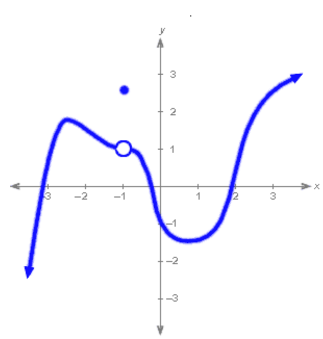
I. $[-3,3]$ II. $[-1,\infty)$ III. $\mathbb{R}$ IV. $[0,17.3]$

Answer: I. No. II. Yes. III. No. IV. Yes.

#3 Given $f(x)=\dfrac{-x}{2+x^n}$, what conditions can be placed on *n* such that *f* will be continuous over $\mathbb{R}$.

Answer: If *n* is a positive even integer, then *f* is continuous over $\mathbb{R}$.

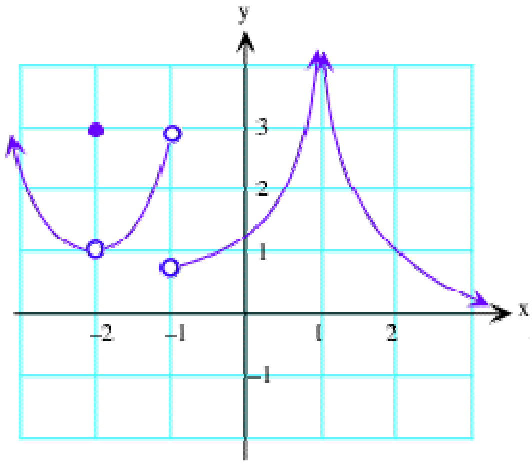
#4 Which condition(s) of continuity are not met by $\beta(x)$ whose graph is given below.



Answer: Condition three is not met. β is discontinuous because $\lim_{x\to a} f(x)\neq f(a)$.

Example Exercise 1

Consider the graph of $f(x)$ below.



State the conditions of continuity that are not met at each discontinuous point within the interval

The graph shows that the curve is discontinuous at $x = -2$ where the function fails to meet the definition of continuity, namely, $f(-2) \neq \lim_{x\to -2} f(x)$.

The graph shows that the curve is discontinuous at $x = -1$ because $\lim_{x\to -1} f(x)$ does not exist and because $f(-1)$ is undefined. For the same reasons, the curve is discontinuous at $x = 1$.

Example Exercise 2

Sketch function f such that f has the following characteristics.

i) f is discontinuous when $x = -5$, $x = -3$, and $x = 2$

ii) $f(-5) = 1$

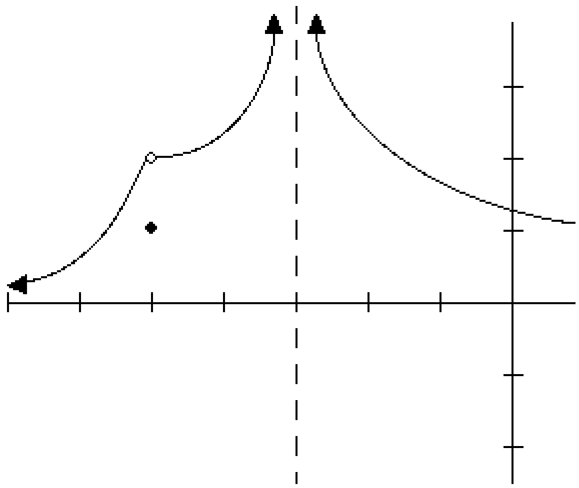
iii) $\lim_{x \to -5} f(x) = 2$

iv) $f(-3)$ is undefined

v) $\lim_{x \to -3} f(x) = \infty$

vi) $\lim_{x \to -\infty} f(x) = 0$

Sketches may vary, but the following graph exhibits all the given characteristics.



Application Exercise

The function $F(r)$ gives the gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet.

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

In the formula, M, R, and G are non-zero constants representing the mass of Earth, the radius of Earth, and the gravitational constant respectively. Is F a D-continuous? In other words, is F continuous for non-zero values of r?