

## Integration by Parts

Integration by substitution reverses the chain rule. *Integration by parts* reverses the product rule. Consider the product rule below.

$$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Integrating both sides of the product rule gives us:

$$\int \frac{d}{dx}[f(x)g(x)] dx = \int [f(x) \cdot g'(x) + g(x) \cdot f'(x)] dx$$

$$f(x)g(x) = \int f(x) \cdot g'(x) dx + \int g(x) \cdot f'(x) dx.$$

Solving for  $\int f(x) \cdot g'(x) dx$  yields the formula below.

The formula for *integration by parts*:

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x) dx$$

Using substitution, the rule can be rewritten as below.

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

As any mediocre witch will tell us, the success of any formula depends entirely on the secret ingredients. Witches are partial to eye of newt and toe of frog, but the secret ingredient for the integration by parts formula is LIATE. What is LIATE? A special coffee? No, it's a special order of functions as below.

**L**ogarithmic functions

**I**nverse trigonometric functions

**A**lgebraic functions

**T**rigonometric functions

**E**xponential functions

The higher a type of function appears on this list, the more likely it should serve as  $f$  in the integration by parts formula. Conversely, the lower a type of function appears on this list, the more likely it should serve as  $g'$ .

Now that we have the formula and the secret ingredient, let's try our luck integrating  $\int x \ln(x) dx$ . Notice that we have two functions,  $x$  and  $\ln(x)$ . Since  $\ln(x)$  is a logarithmic

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function (which appears at the top of our secret ingredient), we let  $f(x) = \ln x$ , which makes  $g'(x) = x$ . We will differentiate  $f(x)$  and integrate  $g'(x)$  as follows.

$$\begin{aligned} f(x) &= \ln(x) & g'(x) &= x dx \\ f'(x) &= \frac{1}{x} dx & g(x) &= \frac{1}{2} x^2 \end{aligned}$$

Now, we can integrate using the formula:

$$\begin{aligned} \int f(x) \cdot g'(x) dx &= f(x) \cdot g(x) - \int g(x) \cdot f'(x) dx \\ \int x \ln(x) dx &= \ln(x) \cdot \frac{1}{2} x^2 - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx \\ &= \frac{1}{2} x^2 \cdot \ln(x) - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \cdot \ln(x) - \frac{1}{2} \cdot \frac{1}{2} x^2 + C \\ &= \frac{1}{2} x^2 \cdot \ln(x) - \frac{1}{4} x^2 + C. \end{aligned}$$

At first glance  $\int \ln(x) dx$  seems too simple for the parts formula since we see only one function,  $\ln(x)$ . Seeing only one function, however, reveals a serious lack of imagination, for  $\ln(x) = 1 \cdot \ln x$ . Thus, we have two functions: a logarithmic function,  $f(x) = \ln(x)$ , and an algebraic function, the constant function,  $g'(x) = 1$ . We differentiate  $f(x)$  and integrate  $g'(x)$  as below.

$$\begin{aligned} f(x) &= \ln(x) & g'(x) &= 1 \\ f'(x) &= \frac{1}{x} & g(x) &= x \end{aligned}$$

Now, integrate using the formula:

$$\begin{aligned} \int 1 \cdot \ln(x) dx &= \ln(x) \cdot x - \int x \cdot \frac{1}{x} dx \\ &= x \cdot \ln(x) - \int 1 dx \\ &= x \cdot \ln(x) - x + C. \end{aligned}$$

Sometimes integration by parts works after a repetition of the formula. Other times it seems parts takes a full circle. Consider  $\int e^x \sin(x) dx$ . Let  $f(x) = \sin(x)$  and  $g'(x) = e^x$ . Differentiate  $f(x)$  and integrate  $g'(x)$  as below.

$$\begin{aligned} f(x) &= \sin(x) & g'(x) &= e^x \\ f'(x) &= \cos(x) & g(x) &= e^x \end{aligned}$$

Integrate by parts:

$$\int e^x \cdot \sin(x) dx = \sin(x) \cdot e^x - \int e^x \cdot \cos(x) dx$$

Start again in order to calculate  $\int e^x \cdot \cos(x) dx$ . Let  $f_2(x) = \cos(x)$  and  $g_2(x) = e^x$ .

Differentiate  $f(x)$  and integrate  $g'(x)$  as below.

$$\begin{aligned} f(x) &= \cos(x) & g'(x) &= e^x \\ f'(x) &= -\sin(x) & g(x) &= e^x \end{aligned}$$

Return to  $\int e^x \cdot \sin(x) dx = \sin(x) \cdot e^x + \int e^x \cdot \cos(x) dx$  and use perform the integration on the right side of the equation using parts.

$$\begin{aligned} \int e^x \cdot \sin(x) dx &= \sin(x) \cdot e^x - \int e^x \cdot \cos(x) dx \\ &= \sin(x) \cdot e^x - \left[ \cos(x) \cdot e^x - \int e^x \cdot -\sin(x) dx \right] \\ &= \sin(x) \cdot e^x - \left[ \cos(x) \cdot e^x + \int e^x \cdot \sin(x) dx \right] \\ &= \sin(x) \cdot e^x - \cos(x) \cdot e^x - \int e^x \cdot \sin(x) dx \end{aligned}$$

It seems we have come full circle, since we have on the right side of the equation an integral equal to the integral on the left except for the negative sign in front. Time for some algebra sleight of hand: let's add the integral to both sides of the equation.

$$\begin{aligned} \int e^x \cdot \sin(x) dx &= \sin(x) \cdot e^x - \cos(x) \cdot e^x - \int e^x \cdot \sin(x) dx \\ \int e^x \cdot \sin(x) dx + \int e^x \cdot \sin(x) dx &= \sin(x) \cdot e^x - \cos(x) \cdot e^x \\ 2 \int e^x \cdot \sin(x) dx &= \sin(x) \cdot e^x - \cos(x) \cdot e^x \end{aligned}$$

Voila! Dividing by two on both sides of the equation finishes the problem.

$$\begin{aligned} \frac{\cancel{2} \int e^x \cdot \sin(x) dx}{\cancel{2}} &= \frac{\sin(x) \cdot e^x - \cos(x) \cdot e^x}{2} \\ \int e^x \cdot \sin(x) dx &= \frac{1}{2} e^x [\sin(x) - \cos(x)] + C \end{aligned}$$

**Practice Problems in *Calculus: Concepts and Contexts* by James Stewart**

1st ed. problem set: Section 5.6 #1–23 odd

2nd ed. problem set: Section 5.6 #3–23 odd

3rd ed. problem set: Section 5.6 #3–21 odd

**Practice Problems in *Calculus: Early Transcendentals* by Briggs and Cochran**

1st ed. problem set: Section 7.1 #1-35 odd

**Possible Exam Problems**

#1 Calculate  $\int x \sin(x) dx$ .

Answer:  $\int x \sin(x) dx = -x \cos(x) + \sin(x) + C$

#2 Evaluate  $\int_1^4 \sqrt{x} \ln(x) dx$ .

Answer:  $\int_1^4 \sqrt{x} \ln(x) dx = \frac{16}{3} \ln(4) - \frac{28}{9}$

**Example Exercise**

Evaluate  $\int (\tan^{-1} x) dx$ .

Let  $f(x) = \tan^{-1} x$  and let  $g'(x) = 1$ . Whence,  $f'(x) = \frac{1}{x^2 + 1}$  and  $g(x) = x$ . Substitute into the parts formula,  $\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$  as below.

$$\int (\tan^{-1} x) dx = x \cdot \tan^{-1} x - \int \frac{1}{x^2 + 1} \cdot x dx$$

Let  $u = x^2 + 1$  so that  $du = 2x dx$  and  $0.5 du = x dx$ . Now, we can substitute as below.

$$x \cdot \tan^{-1} x - \int \frac{1}{x^2 + 1} \cdot x dx = x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{1}{u} du$$

Integrating, we have the following.

$$x \cdot \tan^{-1} x - \int \frac{1}{x^2 + 1} \cdot x dx = x \cdot \tan^{-1} x - \frac{1}{2} \ln |u| + C$$

Simplifying with log properties and substituting  $x^2 + 1$  for  $u$  obtains the final answer.

$$\int (\tan^{-1} x) dx = x \cdot \tan^{-1} x - \ln(\sqrt{x^2 + 1}) + C$$

**Application Exercise**

A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is  $m$ , the fuel is consumed at rate  $r$ , and exhaust gases are ejected with constant  $v_e$  (relative to the rocket). The equation below represents a model for the velocity of the rocket at time  $t$  seconds where  $g$  is the acceleration due to gravity and  $t$  is not too large.

$$v(t) = -gt - v_e \ln\left(1 - \frac{r}{m}t\right)$$

Recall  $g$ ,  $m$ ,  $r$ , and  $v_e$  are constants and find a function for the height of the rocket one minute after liftoff.