Integration by Substitution

In the previous lectures, we learned that differentiation and integration are inverse operations. Since we can differentiate functions without much trouble, we might expect that we can integrate most any function. Alas, this is not so. Many functions escape integration. Nevertheless, there are techniques that can help in particular situations. The lecture presents one such technique called *integration by substitution*.

Consider $\int e^{x^4} \cdot 4x^3\, dx$. Envision the integrand as $\int e^{f(x)} \cdot f'(x)\, dx$. Let $u = f(x)$, then the differential du equals $f'(x)\,dx$, and we have the following.

$$u = x^4$$
$$du = 4x^3 dx$$

Now, we substitute *u* and the differential *du* into the integrand and integrate.

$$\int e^u\, du = e^u + C$$

Lastly, we re-substitute for u so the integral is in terms of the original variable, *x*.

$$\int e^{x^4} \cdot 4x^3\, dx = e^{x^4} + C$$

The difficulty with this method lies in the magic "envisioning" of the integrand in such a way that a nifty substitution of *u* and *du* reduces the integration to a known integral. The method works because it reverses the chain rule:

$$\int f'(g(x))\, g'(x)\, dx = \int f(g(x))'\, dx = f(g(x)) + C\,.$$

If we write $u'(x)\,dx = \frac{du}{dx}dx = du$, then the equation becomes $\int f'(u(x))\,du = f(u(x)) + C$. If the reduction of *dx*'s seems fishy, good. The notation $\frac{du}{dx}$ stands for $u'(x)$, but treating $\frac{du}{dx}$ as a fraction is sometimes ok. Lucky for us, this is one of those times.

In practice, all this fuss with notation is superfluous. Consider $\int 7x^2 \sqrt{1+x^3}\, dx$. Note that the radicand has a derivative that is a scalar multiple of the function outside the radical. This is the "envisioning" step. If we recognize $7x^2$ as a scalar multiple of the derivative of $7+x^3$, we are already adept at integration by substitution. Let $u = 7 + x^3$ so that $u' = 3x^2$, which allows us to write the following (thinking of the du/dx notation as a fraction):

$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2 dx.$$

We want to substitute for $7x^2$, so we multiply *du* by 7/3 as below.

$$\frac{7}{3}\cdot du = \frac{7}{3}\cdot 3x^2 dx$$
$$\frac{7}{3}\cdot du = 7x^2 dx$$

Substituting *u* and $\frac{7}{3}\cdot du$ into the integrand, we have:

$$\int 7x^2\sqrt{1+x^3}\,dx = \int \sqrt{1+x^3}\cdot 7x^2\,dx = \int \sqrt{u}\cdot\frac{7}{3}du\,,$$

which we can integrate as follows:

$$\int \sqrt{u}\cdot\frac{7}{3}du = \frac{7}{3}\cdot\int u^{\frac{1}{2}}du = \frac{7}{3}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{14}{9}\sqrt{u^3} + C = \frac{14}{9}u\sqrt{u} + C\,.$$

Lastly, we re-substitute to replace *u*, so that we have successfully integrated.

$$\int 7x^2\sqrt{1+x^3}\,dx = \frac{14}{9}\left(1+x^3\right)\sqrt{1+x^3} + C$$

Lecture 26

Practice Problems

1st ed. problem set: Section 5.5 #1–31 odd, #37–51 odd
2nd ed. problem set: Section 5.5 #1–31 odd, #37–51 odd
3rd ed. problem set: Section 5.5 #1–33 odd, #39–53 odd

Possible Exam Problems

#1 Calculate $\int \sin^2(x)\cos(x)\,dx$.

Answer: $\int \sin^2(x)\cos(x)\,dx = \frac{1}{3}\sin^3(x) + C$

#2 Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.

Answer: $\int_1^e \frac{\ln x}{x}\,dx = \frac{1}{2}$

Example Exercise

Calculate $\int_0^2 \frac{x}{x+1}\,dx$.

Dividing would simplify the integrand, but this solution will use integration by substitution. Let $u = x+1$. Find the differential of *x* in terms of the differential of *u* as well as *x* in terms of *u*.

$$\begin{aligned} u &= x+1 \\ du &= dx \end{aligned} \quad \text{and} \quad \begin{aligned} u &= x+1 \\ u-1 &= x \end{aligned}$$

Substitute into the integral accordingly. Remember to change the limits of integration to *u*-values.

$$\int_0^2 \frac{x}{x+1}\,dx = \int_1^3 \left[\frac{u-1}{u}\right] du$$

Compute the definite integral as below.

$$\begin{aligned} \int_1^3 \left[\frac{u-1}{u}\right] du &= \int_1^3 \left[1-\frac{1}{u}\right] du \\ &= \left(u - \ln|u|\right)\Big|_1^3 \\ &= \left(3-\ln|3|\right)-\left(1-\ln|1|\right) \\ &= 2-\ln(3) \end{aligned}$$

Hence, $\int_0^2 \frac{x}{x+1}\,dx = 2-\ln(3) \approx 0.9$.

Application Exercise

The Evaluation Theorem says that if *f* is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where *F* is any antiderivative of *f*. We know F' represents the rate of change of $y = F(x)$ with respect to *x*. Clearly, $F(b) - F(a)$ represents the total change in *y* when *x* changes from *a* to *b*. Hence, the definite integral of *a rate of change of a function F* over $[a,b]$ equals the total change in the function *F* over $[a,b]$, i.e., $\int_a^b F'(x)\,dx = F(b) - F(a)$ or $\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$.

Let dp/dt represent the rate of growth of a population *p*. Find the total increase in population from t_1 to t_2 given the particular information below.

$$\frac{dp}{dt} = 900e^{0.09t},\ t_1 = 1,\ t_2 = 3$$