The Definite Integral

This lecture introduces the concept of a *definite integral*. As the name implies, the definite integral is related to the indefinite integral—sort of. Before explaining, we will take time out to tip our hats to the reader who is quietly disturbed by the seeming incongruence of the previous chapter. Examining the flow chart of recent lectures below may make the incongruence apparent.

21. Antiderivative → 22. Indefinite Integral → 23. Riemann Sum → 24. Definite Integral

Since the indefinite integral is simply a new name for the antiderivative, the flow from 21 to 22 seems logical. But, what is the discussion of Riemann sum doing betwixt the indefinite and definite integrals? A Riemann Sum is a sum of the areas of contiguous rectangles that approximate the area between a curve and the *x*-axis. A discussion of areas may seem out of place following a discussion of antiderivatives. The big surprise in calculus, however, resides in the fact that areas beneath curves are indeed related to antiderivatives. Before we elucidate this fact, let's return to the definite integral.

Divorced from any discussion of area, the definite integral can be viewed as a special function with three inputs and one output. Two of the inputs are numbers called the *limits of integration*. These inputs appear at the top and bottom of the integral symbol. For instance, if we write $\int_a^b f(x)\,dx$, then the limits of integration are *a* and *b* where *a* is called the *lower limit of integration* and *b* is called the *upper limit of integration*. The third input is the function $f(x)$ called the *integrand*. To compute the output, we find the indefinite integral with respect to *x*, i.e., we find some $F(x)$ that is an antiderivative of $f(x)$. Then we find the difference of $F(b)$ and $F(a)$.

Consider the definite integral $\int_{-1}^{3} 3x^2\,dx$. The limits of integration are negative one and three. The integrand is $f(x)=3x^2$. An antiderivative of $f(x)$ is $F(x)=x^3$. Calculating the difference of $F(3)$ and $F(-1)$ finds the definite integral; thus, $\int_{-1}^{3} x^2\,dx = 28$ as shown below.

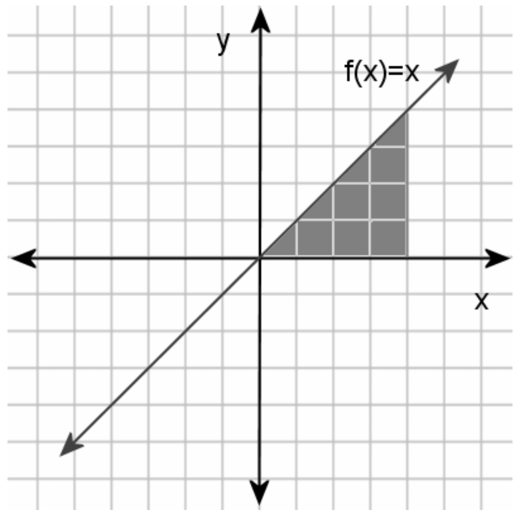
$$\int_{-1}^{3} 3x^2\,dx = x^3 \Big|_{-1}^{3} = (3)^3 - \left[(-1)^3\right] = 27 - [-1] = 27 + 1 = 28.$$

On the right side of the first equal mark above, we have a vertical bar with the limits of integration posted atop and below. This notation is used after integrating and before calculating the difference, $F(b)-F(a)$.

Let's evaluate another definite integral. Consider $\int_0^4 x\,dx$.

$$\int_0^4 x\,dx = \frac{1}{2}x^2 \Big|_0^4 = \frac{1}{2}(4)^2 - \left[\frac{1}{2}(0)^2\right] = \frac{1}{2}\cdot 16 - 0 = 8.$$

Coincidentally (or, at least, seemingly coincidentally), the area between $f(x) = x$ and the *x*-axis along the interval [0,4] equals eight square units, which is the area of the region bounded by the curves, $y = x$, $y = 0$, $x = 0$, and $x = 4$ as in the figure below.



It is a surprising fact, that for continuous functions the definite integral equals the net area between the curve and the *x*-axis along an interval [*a*,*b*] where *a* and *b* are the lower and upper limits of integration respectively. Indeed, we recall from Lecture 23 that the net area under a continuous function *f* is defined as a special limit of a Riemann sum. We will now define this special limit as the *definite integral* as below.

Assume that *f* is a continuous function along the interval $[a,b]$. Then the *definite integral* of *f* with respect to *x* from *a* to *b* is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where *n* represents the number of approximating rectangles with a uniform width equal to $\Delta x = (b-a)/n$ and where x_i represents some point in the *i*th subinterval of $[a,b]$.

At this juncture, it is certainly fair to note that we have two ways of interpreting the definite integral. Indeed we do. We first say $\int_a^b f(x)\,dx = F(b) - F(a)$, then we say $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. For both statements to be valid we must be able to show the following.

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

First, we note the partitioning of the interval $[a,b]$ into n subintervals with endpoints $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$ where $x_0 = a$ and $x_n = a$. Using this partitioning, we can write Equation 1 below.

Equation 1: $F(b)-F(a)=F(x_n)-F(x_{n-1})+F(x_{n-1})-F(x_{n-2})+\cdots+F(x_1)-F(x_0)$

F is continuous because it is differentiable, so we can apply the Mean Value Theorem (from Lecture 15) to F on each subinterval $[x_{k-1},x_k]$. Thus, there exists a number x_i between x_{k-1} and x_k such that

$$F(x_k)-F(x_{k-1})=F'(x_i)(x_i-x_{i-1}).$$

Since $F'(x_i)=f(x_i)$ and $x_i-x_{i-1}=\Delta x$, we have:

$$F(x_k)-F(x_{k-1})=f(x_i)\Delta x.$$

Returning to Equation 1, we can substitute for each $F(x_k)-F(x_{k-1})$ as below.

$$F(b)-F(a)=F(x_n)-F(x_{n-1})+F(x_{n-1})-F(x_{n-2})+\cdots+F(x_1)-F(x_0)$$
$$F(b)-F(a)=f(x_n)\Delta x+f(x_{n-1})\Delta x+\cdots+f(x_1)\Delta x$$

The right side can be simplified using sigma notation.

$$F(b)-F(a)=f(x_n)\Delta x+f(x_{n-1})\Delta x+\cdots+f(x_1)\Delta x$$
$$F(b)-F(a)=\sum_{i=1}^{n} f(x_i)\Delta x$$

We now take the limit as n approaches infinity of each side, noting that the left side is a constant.

$$\lim_{n\to\infty}\left[F(b)-F(a)\right]=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$F(b)-F(a)=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Thus, we have proven the *Evaluation Theorem*.

The *Evaluation Theorem*: If f is continuous on the interval $[a,b]$ then

$$\int_a^b f(x)\,dx=F(b)-F(a) \quad \text{where } F'(x)=f(x).$$

Now that we have the Evaluation Theorem, we will present some basic properties of definite integrals.

The Basic Rules for Definite Integrals

Suppose *f* and *g* are continuous functions and *c* is any constant.

1. $\int_a^b c\,dx = c(b-a)$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

4. $\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If $a \le c \le b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

6. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Practice Problems

1st ed. problem set:	Section 5.3 #7–29 odd
2nd ed. problem set:	Section 5.3 #9–33 odd
3rd ed. problem set:	Section 5.3 #1–27 odd

Possible Exam Problems

#1 Evaluate $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx$.

Answer: $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1} \right) dx = -4 + 3\tan^{-1}(2) \approx -0.67855$

#2 Calculate $\int_{-2}^{1} |x|\, dx$.

Answer: $\int_{-2}^{1} |x|\, dx = \int_{-2}^{0} (-x)\, dx + \int_0^1 x\, dx = 2.5$

#3 If R equals the area of the region between the x-axis and the curve traced by $f(x)$ along the interval [0,2π], which of the following expressions equals R assuming $f(x) = \sin x$?

A. $R = \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx$

B. $R = \int_0^{\pi} \sin x\, dx + \int_{2\pi}^{\pi} \sin x\, dx$

C. $R = \int_0^{\pi} \sin x\, dx + \left| \int_{\pi}^{2\pi} \sin x\, dx \right|$

D. $R = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx$

E. All of the above.

Answer: E. All of the above.

Example Exercise 1

Evaluate $\int_{-1}^{3}\left(x^3-5x^2+3x+9\right)dx$.

$$\int_{-1}^{3}\left(x^3-5x^2+3x+9\right)dx=\frac{1}{4}x^4-\frac{5}{3}x^3+\frac{3}{2}x^2+9x\,\Big|_{-1}^{3}$$

$$=\frac{1}{4}(3)^4-\frac{5}{3}(3)^3+\frac{3}{2}(3)^2+9(3)-\left(\frac{1}{4}(-1)^4-\frac{5}{3}(-1)^3+\frac{3}{2}(-1)^2+9(-1)\right)$$

$$=\frac{81}{4}-\frac{135}{3}+\frac{27}{2}+27-\left(\frac{1}{4}+\frac{5}{3}+\frac{3}{2}-9\right)$$

$$=\frac{243}{12}-\frac{540}{12}+\frac{162}{12}+\frac{324}{12}-\left(\frac{3}{12}+\frac{20}{12}+\frac{18}{12}-\frac{108}{12}\right)$$

$$=\frac{189}{12}-\left(-\frac{67}{12}\right)$$

$$=\frac{256}{12}$$

$$\int_{-1}^{3}\left(x^3-5x^2+3x+9\right)dx=\frac{64}{3}$$

Example Exercise 2

Evaluate $\int_{\sqrt{3}/3}^{1}\left(\frac{1}{x^2+1}\right)dx$.

$$\int_{\sqrt{3}/3}^{1}\left(\frac{1}{x^2+1}\right)dx=\tan^{-1}(x)\Big|_{\sqrt{3}/3}^{1}=\tan^{-1}(1)-\tan^{-1}\left(\sqrt{3}/3\right)=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$$

Example Exercise 3

Evaluate $\int_0^3 |2x-3|dx$.

Write the integrand as a piece-wise function as below.

$$f(x)=\begin{cases}-(2x-3) & \text{if } x<3/2 \\ 2x-3 & \text{if } x\geq 3/2\end{cases}$$

The graph of $f(x)$ makes it apparent that $\int_0^{3/2}\left[-(2x-3)\right]dx=\int_{3/2}^{3}[2x-3]dx$; therefore, $\int_0^3|2x-3|dx=2\cdot\int_0^{3/2}\left[-(2x-3)\right]dx$ as given below.

$$\int_0^3|2x-3|dx=2\cdot\int_0^{3/2}\left[-(2x-3)\right]dx$$

$$\int_0^3|2x-3|dx=2\cdot\int_0^{3/2}\left[-2x+3\right]dx$$

$$\int_0^3|2x-3|dx=2\cdot\left[-x^2+3x\right]_0^{3/2}$$

$$\int_0^3|2x-3|dx=2\cdot\left[-\left(\frac{3}{2}\right)^2+3\cdot\frac{3}{2}-\left(-0^2+3\cdot 0\right)\right]$$

$$\int_0^3|2x-3|dx=2\cdot\left[-\frac{9}{4}+\frac{9}{2}\right]$$

$$\int_0^3|2x-3|dx=2\cdot\left[\frac{9}{4}\right]$$

$$\int_0^3|2x-3|dx=\frac{9}{2}$$

Application Exercise

The Evaluation Theorem says that if *f* is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where *F* is any antiderivative of *f*. We know F' represents the rate of change of $y = F(x)$ with respect to *x*. Clearly, $F(b) - F(a)$ represents the total change in *y* when *x* changes from *a* to *b*. Hence, the definite integral of *a rate of change of a function F* over $[a,b]$ equals the total change in the function *F* over $[a,b]$, i.e., $\int_a^b F'(x)\,dx = F(b) - F(a)$ or $\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$.

Let dp/dt represent the rate of growth of a population *p*. Write a formula for the total change in *p* over the interval $[t_1, t_2]$.

The Fundamental Theorem of Calculus

The previous lecture presented the Evaluation Theorem. Here, we will incorporate the Evaluation Theorem as part of the *Fundamental Theorem of Calculus*.

Consider the equation $g(x)=\int_a^x f(t)\,dt$. Differentiating each side we have:

$$g'(x)=\frac{d}{dx}\int_a^x f(t)\,dt.$$

Employing the Evaluation Theorem on the right, we obtain the following.

$$g'(x)=\frac{d}{dx}\left[F(x)-F(a)\right]$$
$$g'(x)=\frac{d}{dx}\left[F(x)\right]-\frac{d}{dx}\left[F(a)\right].$$

Since $F(a)$ is a constant, we can simplify to

$$g'(x)=\frac{d}{dx}\left[F(x)\right].$$

By the Evaluation Theorem, $\frac{d}{dx}\left[F(x)\right]=f(x)$. Thus,

$$g'(x)=f(x).$$

This conclusion incorporated with the Evaluation Theorem is called the *Fundamental Theorem of Calculus*.

The *Fundamental Theorem of Calculus*: Suppose f is continuous on $[a,b]$.

1. If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.
2. $\int_a^b f(x)\,dx=F(b)-F(a)$ where $F'(x)=f(x)$.

The Fundamental Theorem of Calculus allows us to think of integrals or antiderivatives as net areas. The second part of the theorem, which deals with the definite integral, shows us how to find net areas for defined regions. The first part of the theorem allows us to think of indefinite integrals as an area function, that is, a function whose outputs equal a net area under the curve from some given constant to the input.

Practice Problems

1st ed. problem set:	Section 5.4 #11–13 odd, #26b
2nd ed. problem set:	Section 5.4 #7–9 odd, #22b
3rd ed. problem set:	Section 5.4 #7–9 odd. #24b

Possible Exam Problems

#1 If $g(x) = \int_0^x (2+\cos t)\,dt$, find $g'(x)$.

Answer: $g'(x) = 2 + \cos x$

#2 Let $A(x) = \int_0^x e^t dt$. What application does $A(x)$ have?

Answer: $A(x)$ gives the area between the positive x-axis and the curve $y = e^x$

Example Exercise

Let $R(x) = \int_0^x t\,dt$. Evaluate $R(12)$.

If $R(x) = \int_0^x t\,dt$, then $R(12) = \int_0^{12} t\,dt$ as below.

$$R(12) = \int_0^{12} t\,dt$$

$$R(12) = \frac{1}{2}t^2 \Big|_0^{12}$$

$$R(12) = \frac{1}{2}(12)^2 - \frac{1}{2}(0)^2$$

$$R(12) = \frac{1}{2}\cdot 144$$

$$R(12) = 72$$

Application Exercise

In 1819, French physicist Augustin-Jean Fresnel ("fray nell") used mathematical models to describe the diffraction of light waves. The function below represents one of these models now called a Fresnel function.

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

Use the Fundamental Theorem of Calculus to find $S'(x)$.