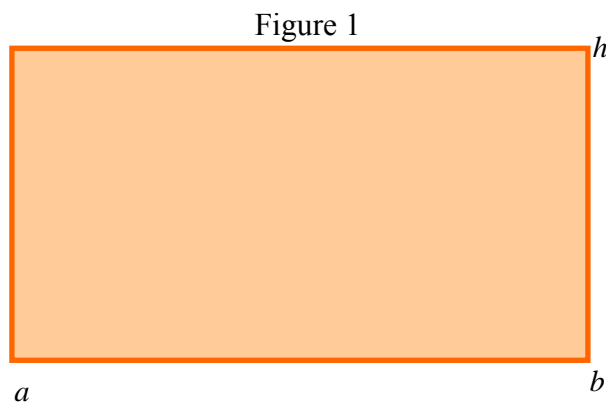


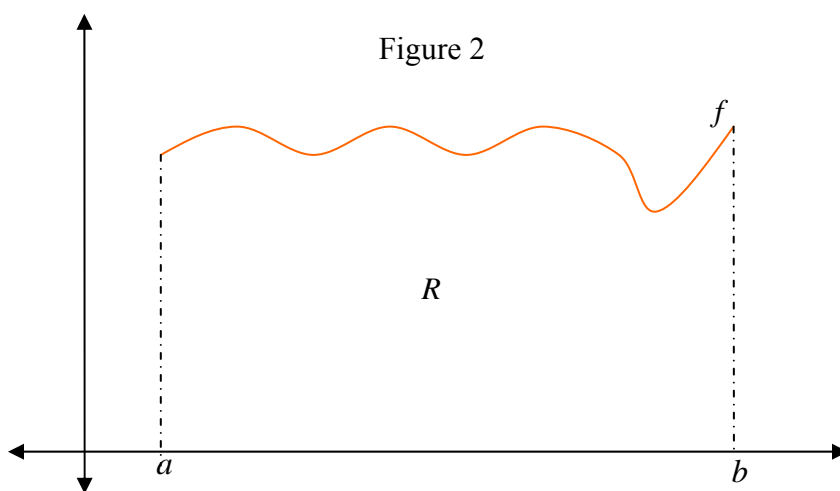
### The Area Problem and the Riemann Sum

Recall that calculus is a branch of mathematics derived from two problems: finding instantaneous rate of change and the area between a curve and the  $x$ -axis. In previous lectures, we have seen that the derivative gives the instantaneous rate of change. In the forthcoming lectures, we will see that the antiderivative or integral gives the area between a curve and the  $x$ -axis. For now, we will consider techniques for estimating the area under a curve.

We can find the area of rectangles. For instance, finding the area of the rectangle below with height  $h$  and length  $b - a$  simply requires finding the product  $h \cdot (b - a)$ .



If the region is bounded with a curved side, finding the area is not so simple. Consider, for example, the region shown below that is bounded by  $f$ , the  $x$ -axis, and the lines  $x = a$  and  $x = b$ .



We see that computing the exact area of  $R$ , the region between  $f$  and the  $x$ -axis, is a daunting task. Despite the apparent difficulty of computing the *exact* area of  $R$ , superimposing a rectangle over  $R$  as in Figure 3 reveals a simple process for computing an *estimate* for the area.

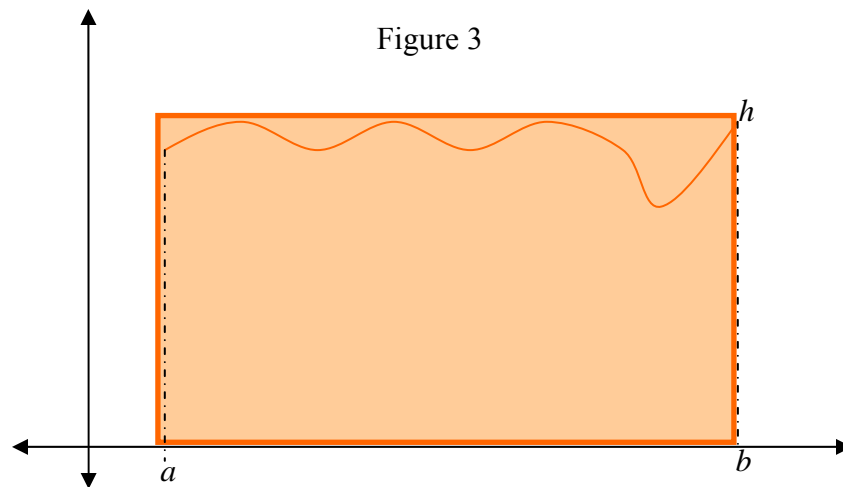
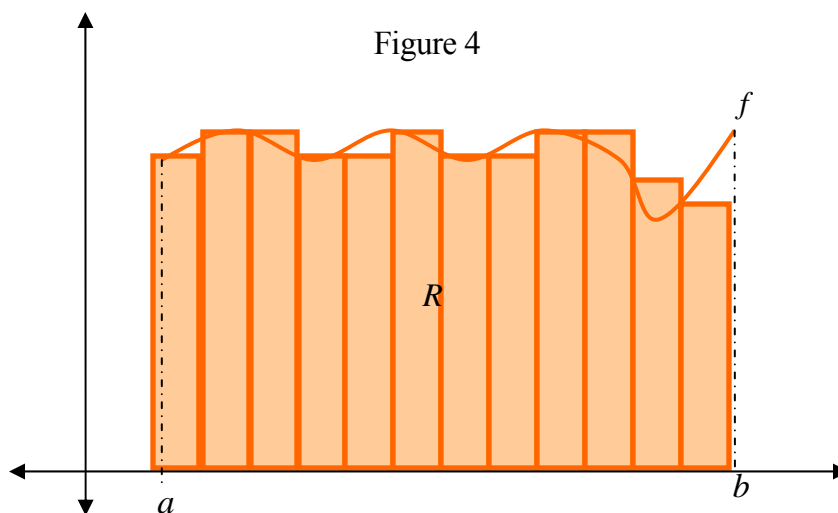


Figure 3 shows that the product  $h \cdot (b - a)$  provides a gross yet reasonable estimate for the area in  $R$ . If  $A$  equals the area of  $R$ , then clearly  $h \cdot (b - a) > A$  since  $h$  is greater than or equal to all values of  $f$  along the interval  $[a, b]$ .

We can obtain a better estimate, however, if we divide the rectangle with height  $h$  into  $n$  rectangles of some uniform width equal to  $(b - a)/n$  with heights determined in some manner by  $f$ . In Figure 4 below, each rectangle intersects  $f$  at the top left corner so that the height of each rectangle is determined by  $f(x_i)$  where  $x_i$  corresponds to the left end point of the subintervals of  $[a, b]$  that comprise the widths of the rectangles. The sum of the areas of the rectangles provides an improved estimate of the area of  $R$ .



The number of rectangles used to generate an estimate is arbitrary, but we see intuitively that the greater the number of rectangles used the more accurate the estimate will be. Similarly, the choice to determine the height of each rectangle using  $f(x_i)$  where  $x_i$  corresponds to the left end point of the subintervals of  $[a, b]$  is also arbitrary. The points  $x_i$  could refer to the right

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end points of the subintervals of  $[a, b]$ , the midpoints of the subintervals of  $[a, b]$ , or even just any random point within each subinterval of  $[a, b]$ .

Returning to Figure 4, we will now attempt to write an approximation formula for  $A$ , the area of  $R$ . Since the number of rectangles used for the estimate is arbitrary, let's represent the number of rectangles with  $n$ . Recall that the rectangles span  $[a, b]$ , and each rectangle has a width given by  $(b - a)/n$ . We will let  $x_1, x_2, \dots, x_n$  be sample points taken from the first, second, . . . , and  $n$ th subintervals. Accordingly, we have the following.

$$A \approx \text{sum of the areas of the } n \text{ rectangles of width } \frac{b-a}{n}$$
$$A \approx \frac{b-a}{n} [f(x_1) + f(x_2) + \dots + f(x_n)]$$

The notation above is not bad if  $n$  is a tame number like four, but if  $n$  is large, the notation becomes somewhat long-winded. Accordingly, we employ sigma notation. The Greek letter sigma,  $\sum$ , indicates the summation of terms. Letting  $x_i$  be some arbitrary point within each closed subinterval of  $[a, b]$ , and letting  $\Delta x$  represent the width of each rectangle,  $(b - a)/n$ , we obtain the following.

$$A \approx \sum_{i=1}^n f(x_i) \Delta x$$

The letter  $i$  is called the index of summation and stands for the  $i$ th (that is, the first, second, . . . ,  $n$ th) subinterval. The sum  $\sum_{i=1}^n f(x_i) \Delta x$  is called a *Riemann sum*. Formally, we define a Riemann sum as below.

A *Riemann sum*, denoted  $\sum_{i=1}^n f(x_i) \Delta x$ , is the sum of  $n$  contiguous rectangles along the interval  $[a, b]$  with bases of uniform width,  $\Delta x$ , equal to  $(b - a)/n$  and with heights,  $f(x_i)$ , where  $x_i$  is some arbitrary point within each closed subinterval represented by the bases.

A *Riemann sum*, therefore, is a sum of  $n$  contiguous rectangles along an interval  $[a, b]$  with uniform positive widths and with heights determined by arbitrary values of a function. In Figure 4, the function  $f$  is positive, so all the heights of the contiguous rectangles take positive values. Positive height values multiplied by positive width values give positive area values, and the sum of these area values approximates the area between the  $x$ -axis and  $f$ . If, however,  $f$  were to take some negative values along the interval  $[a, b]$ , then some of the heights would be assigned negative values. Negative height values multiplied by positive width values give negative area values. A negative area represents the fact that the particular rectangle is below the  $x$ -axis (since its height is given by a negative value of  $f$ ). Accordingly, if a function  $f$  along an interval  $[a, b]$

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takes both positive and negative values, then the corresponding Riemann sum is the sum of the areas of the rectangles that lie above the  $x$ -axis and the *negatives* of the areas of the rectangles that lie below the  $x$ -axis (the areas of the rectangles above the  $x$ -axis *minus* the areas of the rectangles below the  $x$ -axis). In other words, if  $f$  is continuous and takes positive and negative values along  $[a,b]$ , then any corresponding Riemann sum approximates a *net area* where the net area equals the difference between the area of regions above the  $x$ -axis but under  $f$  and the area of regions below the  $x$ -axis but above  $f$ .

With a Riemann sum approximation in mind, we will define the exact net area between  $f$  and the  $x$ -axis as the limit of the sum of the areas of approximating rectangles as the number of rectangles approach infinity ( $n \rightarrow \infty$ ).

Assume that  $f$  is a continuous function along the interval  $[a,b]$ . The net area  $A$  of the region bounded by  $f$  and the  $x$ -axis is given by

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where  $n$  represents the number of approximating rectangles with a uniform width equal to  $\Delta x = (b - a)/n$  and where  $x_i$  represents some point in the  $i$ th subinterval of  $[a,b]$ .

If  $f$  is positive over the interval of interest, then  $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$  gives the area between  $f$  and the  $x$ -axis. If  $f$  takes positive and negative values over the interval, then  $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$  gives a net area, which is the area above the  $x$ -axis but under  $f$  minus the area below the  $x$ -axis but above  $f$ .

### Practice Problems

- 1st ed. problem set: Section 5.1 #1, #3a also Section 5.2 #1, #2  
2nd ed. problem set: Section 5.1 #1, #3a also Section 5.2 #1–4 all  
3rd ed. problem set: Section 5.1 #1, #3a also Section 5.2 #1–4 all

### Possible Exam Problems

- #1 A Riemann sum refers to the sum of what type objects/quantities?  
a) heights  
b) areas  
c) functions  
d) subintervals  
e) arbitrary points

Answer: b) areas

- #2 Use a Riemann sum to estimate the area between  $f$  and the  $x$ -axis along the interval  $[1, 3]$  for  $f(x) = 0.25x^3$ .

Answer: Using four rectangles and right end points as the sample point in each subinterval:

$$A \approx \sum_{i=1}^n f(x_i) \Delta x$$

$$A \approx \sum_{i=1}^4 f(x_i) \cdot \frac{3-1}{4}$$

$$A \approx \sum_{i=1}^4 f(x_i) \cdot 0.5$$

$$A \approx 0.5 \cdot \sum_{i=1}^4 f(x_i)$$

$$A \approx 0.5 \left[ 0.25(1.5)^3 + 0.25(2)^3 + 0.25(2.5)^3 + 0.25(3)^3 \right]$$

$$A \approx 0.5[13.5] \approx 6.75 \text{ sq. units}$$

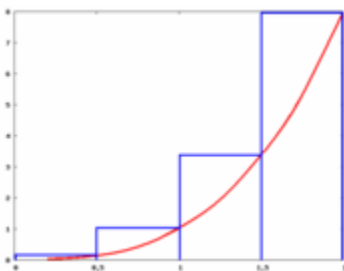
**Example Exercise**

Consider the area between the curve  $y = x^3$  and the  $x$ -axis over the interval  $[0,2]$ . Use a sketches to show how to obtain over and under estimates for the area using Riemann sums.

Let  $y = f(x)$ . Note that the function increases. Arbitrarily select a number of rectangles like four. Determine the width of each rectangle.

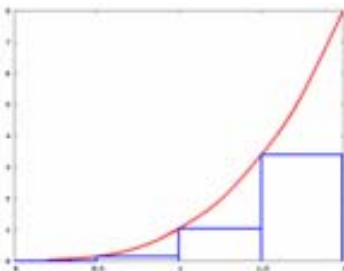
$$w = \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

For an over-estimate, select heights in a manner that too much area is included. A right Riemann sum will do the trick as below.



$$\sum_{i=1}^4 [f(x_i) \cdot \Delta x] = \frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} + \frac{27}{8} \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} = 6.25 \text{ sq. units}$$

For an under-estimate, select heights in a manner that too little area is included. A left Riemann sum will suffice as shown.



$$\sum_{i=1}^4 [f(x_i) \cdot \Delta x] = 0 \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} + \frac{27}{8} \cdot \frac{1}{2} = 2.25 \text{ sq. units}$$

### Application Exercise

Let's concern ourselves with the distance travelled by a object dropped near the earth's surface at  $t = 0$  assuming air resistance is negligible. In the absence of air resistance, all falling bodies accelerate at the same rate. Close to the surface of the earth the gravitational acceleration of a falling body has the constant value  $g \approx 9.81 \text{ m/sec}^2$ . The function describing the acceleration of a moving object equals the derivative of the function describing the object's velocity,  $v(t)$ . Also, the function describing the velocity of a moving object equals the derivative of the function giving the object's position (where position is a distance relative to some arbitrary point).

The table below gives the observed velocity of a falling object.

time interval	velocity in <i>m/sec</i> of the object at left-end of the time interval	velocity in <i>m/sec</i> of the object at right-end of the time interval
[0,0.4]	0.00	3.92
[0.4,0.8]	3.92	7.84
[0.8,1.2]	7.84	11.76
[1.2,1.6]	11.76	15.68
[1.6,2.0]	15.68	19.60

Let  $s(t)$  represent the distance the falling object has traveled. Use a Riemann sum to approximate  $s(2)$ .