Antiderivatives

In the next lecture, we begin doing things backward with the undoing—so to speak—of differentiation. In the next lecture, we will call this undoing process integration. In this lecture, we will call it antidifferentiation. The goal of antidifferentiation is to find antiderivatives (note the plural).

Consider the two functions below.

$$F_1(x) = x^2 + 7x + 12 \quad F_2(x) = x^2 + 7x + 13$$

These two functions F_1 and F_2 have much in common. They are both quadratic functions whose derivatives are equal: $F_1'(x) = F_2'(x) = f(x) = 2x + 7$. Thus, F_1 and F_2 are both called *antiderivatives* of *f*.

An *antiderivative* of *f* is a function *F* such that $F' = f$.

If the reader feels as if he or she is stuck in some Twilight Zone version of the hit game show Jeopardy!, then we are getting somewhere. Antidifferentiation is a lot like knowing the answer but seeking the question.. If the answer is "forty-two," there are numerous possible questions. What is two plus forty? What is six times seven? What is the difference of forty-nine and seven? [Perhaps even, "What is the secret of life, the universe, and everything?"] Obscure allusions aside, an infinite number of questions could be asked such that "forty-two" is the answer. Similarly, when we ask what function has a derivative equal to *f*, there are infinite answers.

Consider the function $g(x) = x^5$. If we think of the power rule of differentiation in reverse, we can find an antiderivative of *g*. Recall that the power rule states "If $f = x^n$, then $f' = nx^{n-1}$." Instead of subtracting from the exponent, adding one and multiplying by a fortuitous scalar (in this case $1/6$ is the appropriate scalar) acquires the antiderivative as below.

$$G(x) = \frac{1}{6}x^{5+1} = \frac{1}{6}x^6$$

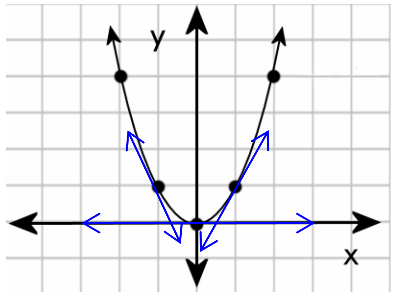
If *G* is an antiderivative of *g*, then $G' = g$ as demonstrated here:

$$G'(x) = 6 \cdot \frac{1}{6}x^{6-1} = x^5 = g(x).$$

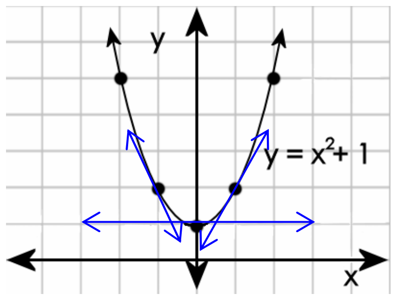
Recall that if *c* is any real number constant and $f = c$, then $f' = 0$. This means any real number can be added to *G* without changing the fact that *G* is an antiderivative of *g*. Generalizing this fact gives us the *Translation Theorem* stated below.

Translation Theorem: If *F* is an antiderivative of *f*, then *F* + *C* is an antiderivative of *f* where C is an arbitrary real number constant.

The Translation Theorem can be shown geometrically. Consider $y = x^2$ graphed below.



It is easy to see that $y = x^2$ has a horizontal tangent line at zero and that all the tangent lines prior to zero would have negative slopes while all the tangent lines after zero will have positive slopes. Now consider $y = x^2 + 1$ graphed blow.



The added constant translates the curve up one unit, but the intervals of the curve's behaviors remain the same. Again, the only horizontal tangent occurs at zero. The tangents for $x < 0$ will all have negative slopes. The tangents for $x > 0$ will all have positive slopes.

Hopefully, the discussion above makes it clear that there are an infinite number of antiderivatives of a function *f*. Sometimes, it is important to find one particular antiderivative of a given function. This is possible if one point on the antiderivative is known. For instance, consider $f(x) = 2x$. What if we wanted to find the antiderivative that passes through the point (0,1)? This type of problem is sometimes called an *initial value problem*. To solve it, we first find the antiderivative in general by thinking of the power rule in reverse.

$$F(x) = \frac{1}{2} \cdot 2x^{1+1} + C$$
$$F(x) = x^2 + C$$

Taking the derivative of *F*, we see that $F' = f$, so we have a general form for the antiderivative of *f*. If we plug in the given *initial values* (0,1), we can solve for *C*.

$$F = x^2 + C$$
$$1 = 0^2 + C$$
$$1 = C$$

Thus, $F(x) = x^2 + 1$ is the particular antiderivative of *f* that contains the point (0,1).

Lecture 21

Practice Problems

1st ed. problem set:	Section 4.9 #1–13 odd
2nd ed. problem set:	Section 4.9 #1–17 odd
3rd ed. problem set:	Section 4.9 #1–17 odd

Possible Exam Problem

If $p(x) = 12x^3 + 6x^2$, how many antiderivatives of *p* exist? What do they look like?

Answer: Function *p* has an infinite number of antiderivatives all of the form $P(x) = 3x^4 + 2x^3 + C$ where *C* is any real number.

Example Exercise 1

Find an antiderivative of $g(x) = 50x^4$.

If *c* is a constant, then $d/dx\left(10x^5 + c\right) = 50x^4$; therefore, $G(x) = 10x^5$ is an antiderivative of $g(x) = 50x^4$.

Example Exercise 2

Find the antiderivative of $f(x) = e^x$ that passes through (0,2).

If *c* is a constant, then $d/dx\left(e^x + c\right) = e^x$. Hence, we are looking for a function of the form $F(x) = e^x + c$ that passes through (0,2).

$$F(x) = e^x + c$$
$$F(0) = e^0 + c$$
$$2 = e^0 + c$$
$$2 = 1 + c$$
$$1 = c$$

$F(x) = e^x + 1$ is an antiderivative of $f(x) = e^x$ that passes through (0,2).

Application Exercise

Rectilinear motion is an idealized form of motion that rarely occurs in actual experience. Nevertheless, rectilinear motion is a simple imaginable type of motion that forms the basis for the analysis of more complicated motions.

Many motions in actual experience are approximately rectilinear. For example, the motion of a high-speed rifle bullet fired horizontally may be essentially rectilinear for a short part of its trajectory (even though its true trajectory is parabolic). Similarly, the motion of a maglev train over a straight stretch of track is essentially rectilinear (maglev trains are high-speed railway trains that travel by magnetic levitation).

Suppose $v(t) = 0.2t + 3$ gives the velocity in feet per second of a maglev train over a straight stretch of track for times $0 < t \leq 120$ with t representing the seconds elapsed since the train left the station. Find the train's distance from the station after one minute.

Indefinite Integral

Lecture 21 discussed antiderivatives. In this section, we introduce new notation and vocabulary. The notation $\int f(x)\,dx$ indicates the general form of the antiderivative of *f* and is called the *indefinite integral*.

The notation $\int f(x)\,dx$ denotes the *indefinite integral*.
The statement $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$.

From the definition above, we see that the notation $\int f(x)dx$ transforms $f(x)$ into the antiderivative. In Lecture 21, this process was called antidifferentiation. Here, we call it *integration*. The $\int$ symbol is called an integral symbol. We recognize the *dx* notation as a differential (the change in *x*), but in the context of integration *dx* indicates that *x* is the variable to be integrated. Suppose we want to integrate a constant like 5, that is, transform 5 into a function (or set of functions) whose derivative equals 5. We call this integration and indicate it by writing $\int 5$, but we need *dx* or *dt* or some indication of the variable. If we write $\int 5\,dx$ then *x* is the variable and $\int 5\,dx = 5x + C$. If we write $\int 5\,dt$ then *t* is the variable and $\int 5\,dt = 5t + C$.

As an operator, the integral operates on functions with respect to a variable, so the argument of integration is a function called the integrand. Moreover, integration transforms the integrand into a function (or set of functions) whose derivative equals the integrand. Like differentiation, integration is a linear operation as defined below.

If *f* and *g* are functions with antiderivatives over some interval *I* and *c* is a scalar, then

$$\int\left[f(x)+g(x)\right]dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Thinking of our rules for differentiation in reverse, we can write some rules for integration. For instance, the power rule of differentiation gives us the rule below.

$$\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C \text{ where } n \neq -1.$$

Specific differentiation facts such as $\frac{d}{dx}\left[e^x\right] = e^x$ convince us (when we think of them in reverse) that the following indefinite integrals are true over some appropriate interval.

Table of Indefinite Integrals

1. $\int x^n \, dx = \frac{1}{n+1} \cdot x^{n+1} + C \quad \text{(where } n \neq -1\text{)}$

2. $\int e^x dx = e^x + C$

3. $\int a^x dx = \frac{a^x}{\ln a} + C$

4. $\int \frac{1}{x} dx = \ln|x| + C$

5. $\int \sin x \, dx = -\cos x + C$

6. $\int \cos x \, dx = \sin x + C$

7. $\int \sec^2 x \, dx = \tan x + C$

8. $\int \csc^2 x \, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int \csc x \cot x \, dx = -\csc x + C$

11. $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

12. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

The rules given in the table above can be verified by differentiating the function on the right side and obtaining the integrand. For example, let's verify the fourth rule in the table.

$$F'(x) = \frac{d}{dx}(-\cos x) + \frac{d}{dx}C$$
$$F'(x) = -\frac{d}{dx}(\cos x) + 0$$
$$F'(x) = -\sin x$$

Practice Problems

1st ed. problem set:	Section 5.3 #39–45 odd
2nd ed. problem set:	Section 5.3 #41–45 odd
3rd ed. problem set:	Section 5.3 #37–41 odd

Possible Exam Problems

#1 The notation $\int f(x)\,dx$ refers to which of the following objects?

a) a number
b) an integrand
c) a set of functions
d) a derivative

Answer: c) a set of functions

#2 If $\int f(t)\,dt = F(t)$, then what special property does $F(t)$ possess?

Answer: $F(t)$ possess the property that $F'(t) = f(t)$.

Example Exercise 1

Evaluate $\int \left(x^4 + 4x\right)dx$.

If $n \neq -1$, then $\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + C$. Apply this rule as below.

$$\int \left(x^4 + 4x\right)dx = \int x^4 dx + 4\int x\,dx = \frac{1}{5}x^5 + 4 \cdot \frac{1}{2}x^2 + C = \frac{1}{5}x^5 + 2x^2 + C$$

Example Exercise 2

Evaluate $\int \left(\sec^2 x + \frac{1}{x^2+1} + e^x + \frac{17}{x} - \sin x\right)dx$.

Apply the rules from the table of indefinite integrals.

$$\int \left(\sec^2 x + \frac{1}{x^2+1} + e^x + \frac{17}{x} - \sin x\right)dx = \tan x + \tan^{-1} x + e^x + 17\ln|x| + \cos x + C$$

Application Exercise

Differential equations appear in several sciences including physics and economics. Integration sometimes solves differential equations of the form below.

$$\frac{dy}{dx} = f(x)$$

The function (or functions) $y(x)$ whose derivative equals $f(x)$ "solves" the equation above. Hence, the general solution of the differential equation $dy/dx = f(x)$ is the indefinite integral below.

$$y(x) = \int f(x)\,dx$$

Use the discussion above to solve the differential equation below. Find a particular solution given that $y(1) = -1$.

$$\frac{dy}{dx} = 2x + 6$$

Application Exercise

Let's concern ourselves with the distance travelled by a object dropped near the earth's surface at $t = 0$ assuming air resistance is negligible. In the absence of air resistance, all falling bodies accelerate at the same rate. Close to the surface of the earth the gravitational acceleration of a falling body has the constant value $g \approx 32\,\text{ft}/\text{sec}^2$. Recall that the function describing the acceleration of a moving object equals the derivative of the function describing the object's velocity, $v(t)$. Recall also that the function describing the velocity of a moving object equals the derivative of the function giving the object's position (where position is a distance relative to some arbitrary point). Let $s(t)$ denote distance the falling object has fallen. Find $s(t)$.