

L'Hospital's Rule

This lecture presents *l'Hospital's Rule* (pronounced "lōp-ī-talls"), which provides a technique for finding limits after direct substitution yields an indeterminate.

L'Hospital's Rule: Let f and g be functions that are differentiable on an open interval I , except possibly at the number a in I and that for all $x \neq a$ in I , $g'(x) \neq 0$. Suppose further that $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ or that $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$. Then it follows that if

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L, \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L \text{ and we write } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \stackrel{H}{=} \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L.$$

The statement is valid if all the limits are right-hand limits or left-hand limits.

Moreover, let f and g be functions that are differentiable for all $x > c$ where c is a positive constant and that for all $x > c$, $g'(x) \neq 0$. Suppose further that $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$ or that $\lim_{x \rightarrow \infty} f(x) = \pm\infty$ and $\lim_{x \rightarrow \infty} g(x) = \pm\infty$. Then it follows that if

$$\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L, \text{ then } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \text{ and we write } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L.$$

The statement is valid if $x \rightarrow \infty$ is replaced by $x \rightarrow -\infty$.

In short, *l'Hospital's Rule* says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that certain conditions are satisfied.

Consider the limit below.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Note that all the conditions for l'Hospital's Rule exist. First, $f(x) = \sin x$ and $g(x) = x$ are both differentiable functions over some interval in this case \mathbb{R} . Second, $g'(x) \neq 0$. Third, $\lim_{x \rightarrow 0} \sin x = 0$ and $\lim_{x \rightarrow 0} x = 0$, which means that $\lim_{x \rightarrow 0} f/g$ is indeterminate. Whenever we face an indeterminate that meets the necessary requirements, we apply l'Hospital's Rule.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$$

Typically, if the limit yields an indeterminate, we look for a way to employ l'Hospital's Rule. Indeterminants were discussed in Lecture 15 and are summarized by the box on the following page.

Indeterminants include indeterminate ratios:

$$1. \frac{0}{0}, \quad 2. \frac{\infty}{\infty};$$

the indeterminate product:

$$3. 0 \cdot \infty;$$

the indeterminate difference:

$$4. \infty - \infty;$$

and the indeterminate powers:

$$5. 0^0, \quad 6. \infty^0, \quad 7. 1^\infty.$$

If the limit yields an indeterminate product, difference or power, it is necessary to algebraically transform the argument of the limit so that the limit yields an indeterminate ratio (see the example with $\lim_{x \rightarrow 0^+} (x+1)^{\cot x}$ below).

Sometimes it takes more than one application of l'Hospital's Rule to find the limit. Consider the limit below.

$$\lim_{x \rightarrow \pi/2^-} \frac{\cos^2 3x}{\cos^2 x}$$

After some examination, we see that $\cos^2 3x \rightarrow 0$ and $\cos^2 x \rightarrow 0$ as $x \rightarrow (\pi/2)^-$, which means we have the indeterminate ratio $0/0$. Accordingly, we apply l'Hospital's Rule.

$$\lim_{x \rightarrow \pi/2^-} \frac{\cos^2 3x}{\cos^2 x} \stackrel{H}{=} \lim_{x \rightarrow \pi/2^-} \frac{-6 \cos 3x \cdot \sin 3x}{-2 \cos x \cdot \sin x} = \lim_{x \rightarrow \pi/2^-} \frac{3(2 \cos 3x \cdot \sin 3x)}{2 \cos x \cdot \sin x} = 3 \lim_{x \rightarrow \pi/2^-} \frac{\sin 6x}{\sin 2x}.$$

After some examination, we see that $\sin 6x \rightarrow 0$ and $\sin x \rightarrow 0$ as $x \rightarrow (\pi/2)^-$, which means we still have the indeterminate ratio $0/0$. Accordingly, we reapply l'Hospital's Rule.

$$3 \lim_{x \rightarrow \pi/2^-} \frac{\sin 6x}{\sin 2x} \stackrel{H}{=} 3 \lim_{x \rightarrow \pi/2^-} \frac{6 \cos 6x}{2 \cos 2x} = 9 \lim_{x \rightarrow \pi/2^-} \frac{\cos 6x}{\cos 2x} = 9 \left(\frac{-1}{-1} \right) = 9$$

Therefore, $\lim_{x \rightarrow \pi/2^-} \frac{\cos^2 3x}{\cos^2 x} = 9$.

Lecture 20

Sometimes we have to think of a way to make l'Hospital's Rule apply to the given limit. Consider the limit below.

$$\lim_{x \rightarrow 0^+} (x+1)^{\cot x}$$

After some examination, we see that $x+1 \rightarrow 1$ and $\cot x \rightarrow \infty$ as $x \rightarrow 0^+$, which means we have the indeterminate power 1^∞ . Accordingly, we look for a way to apply l'Hospital's Rule. If we let $y = (x+1)^{\cot x}$, then $\ln y = \cot x \cdot \ln(x+1)$, and we can write $\ln y = \ln(x+1)/\tan x$ to obtain:

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\tan x},$$

to which we can apply l'Hospital's Rule because $\ln(x+1) \rightarrow 0$ and $\tan x \rightarrow 0$ as $x \rightarrow 0^+$, which means we have the indeterminate ratio $0/0$.

$$\lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x} = 1$$

If $y = (x+1)^{\cot x}$, we know $\lim_{x \rightarrow 0^+} \ln y = 1$, but we want to know $\lim_{x \rightarrow 0^+} y$. Since $e^{\ln y} = y$, we can substitute for y as below.

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^1 = e$$

Practice Problems

1st ed. problem set: Section 4.5 #1–29 odd, #35–37 odd
 2nd ed. problem set: Section 4.5 #5–35 odd, #41–43 odd
 3rd ed. problem set: Section 4.5 #5–39 odd, #45–47 odd,

Possible Exam Problems

#1 Evaluate $\lim_{x \rightarrow 1} \frac{1 - x + \ln x}{x^3 - 3x + 2}$.

Answer: $-\frac{1}{6}$

#2 Given $Q(x) = \frac{\ln(2 + e^x)}{3x}$, show that $Q(x)$ has a horizontal asymptote of $y = \frac{1}{3}$.

Answer: $\lim_{x \rightarrow \infty} \frac{\ln(2 + e^x)}{3x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{2 + e^x} \cdot e^x = \lim_{x \rightarrow \infty} \frac{e^x}{6 + 3e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{3e^x} \lim_{x \rightarrow \infty} \frac{1}{3} = \frac{1}{3}$

#3 Evaluate $\lim_{x \rightarrow 0} \left[(\cos 3x)^{\frac{5}{x}} \right]$.

Answer: 1

Example Exercise 1

$$\text{Evaluate } \lim_{x \rightarrow -1} \left[\frac{x^2 - 1}{x + 1} \right].$$

Direct substitution yields an indeterminate ratio; therefore, L'Hopital's Rule applies.

$$\lim_{x \rightarrow -1} \left[\frac{x^2 - 1}{x + 1} \right] \stackrel{H}{=} \lim_{x \rightarrow -1} \left[\frac{2x}{1} \right] = 2(-1) = -2$$

Example Exercise 2

$$\text{Evaluate } \lim_{x \rightarrow 0^+} \left[\sqrt{x} \cdot \ln(x) \right].$$

Direct substitution yields an indeterminate product, but the argument of the limit can be rewritten so that direct substitution yields an indeterminate ratio.

$$\lim_{x \rightarrow 0^+} \left[\sqrt{x} \cdot \ln(x) \right] = \lim_{x \rightarrow 0^+} \left[x^{\frac{1}{2}} \cdot \ln(x) \right] = \lim_{x \rightarrow 0^+} \left[\frac{\ln(x)}{x^{-\frac{1}{2}}} \right]$$

Now, L'Hospital's Rule applies, so we apply the rule as below.

$$\lim_{x \rightarrow 0^+} \left[\frac{\ln(x)}{x^{-\frac{1}{2}}} \right] \stackrel{H}{=} \lim_{x \rightarrow 0^+} \left[\frac{1/x}{-\frac{1}{2}x^{-\frac{3}{2}}} \right]$$

Simplifying the argument of the limit, we obtain the following.

$$\lim_{x \rightarrow 0^+} \left[\frac{1/x}{-\frac{1}{2}x^{-\frac{3}{2}}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{1}{x} \div \left(-\frac{1}{2x^{\frac{3}{2}}} \right) \right] = \lim_{x \rightarrow 0^+} \left[\frac{1}{x} \left(-2x^{\frac{3}{2}} \right) \right] = \lim_{x \rightarrow 0^+} \left[-2x^{\frac{1}{2}} \right] = \lim_{x \rightarrow 0^+} \left[-2\sqrt{x} \right]$$

Now, direct substitution yields a number.

$$\lim_{x \rightarrow 0^+} \left[-2\sqrt{x} \right] = -2\sqrt{0} = 0$$

Example Exercise 3

Evaluate $\lim_{x \rightarrow \infty} \left[(e^x + x)^{\frac{1}{x}} \right]$.

Direct substitution yields an indeterminate power; therefore, L'Hospital's Rule does not apply immediately. Let $y = (e^x + x)^{\frac{1}{x}}$. Then, we obtain the following.

$$\text{LN}(y) = \text{LN} \left[(e^x + x)^{\frac{1}{x}} \right]$$

$$\text{LN}(y) = \frac{1}{x} \cdot \text{LN} [e^x + x]$$

Now, we write the right side as a fraction and we take the limit of both sides.

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] = \lim_{x \rightarrow \infty} \left[\frac{\text{LN}(e^x + x)}{x} \right]$$

Since the direct substitution gives an indeterminate ratio on the right side, we can apply L'Hospital's Rule. We repeat L'Hospital's Rule until direct substitution does not give an indeterminate ratio.

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] = \lim_{x \rightarrow \infty} \left[\frac{\text{LN}(e^x + x)}{x} \right]$$

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] \stackrel{H}{=} \lim_{x \rightarrow \infty} \left[\frac{e^x + 1}{e^x + x} \right]$$

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] \stackrel{H}{=} \lim_{x \rightarrow \infty} \left[\frac{e^x}{e^x + 1} \right]$$

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] \stackrel{H}{=} \lim_{x \rightarrow \infty} \left[\frac{e^x}{e^x} \right]$$

$$\lim_{x \rightarrow \infty} [\text{LN}(y)] = \lim_{x \rightarrow \infty} (1)$$

Thus, we have $\lim_{x \rightarrow \infty} [\text{LN}(y)] = 1$. Returning to the original problem, we were looking for

$\lim_{x \rightarrow \infty} [y]$. We can rewrite y as $e^{\text{LN} y}$, so we have $\lim_{x \rightarrow \infty} [e^{\text{LN} y}]$. We know that as $x \rightarrow \infty$, $\text{LN} y \rightarrow 1$;

thus, $\lim_{x \rightarrow \infty} [e^{\text{LN} y}] = \lim_{x \rightarrow \infty} [e^1] = e$. So, $\lim_{x \rightarrow \infty} \left[(e^x + x)^{\frac{1}{x}} \right] = e$.

Application Exercise

Engineers who study digital signal processing use the function below, which is called the normalized sinc (pronounced “sink”) function.

$$\text{sinc}(x) = \begin{cases} \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) & \text{if } x = 0 \\ \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0 \end{cases}$$

Use L'Hospital's Rule to evaluate $\text{sinc}(0)$.