

Evaluating Limits with Direct Substitution

To evaluate a limit, the first step requires *direct substitution*. For example, consider the function $f(x) = 5x^3$. To evaluate $\lim_{x \rightarrow -1} f(x)$, we substitute negative one for x as below.

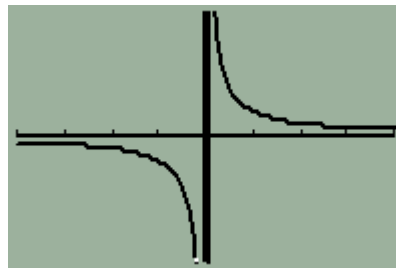
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} [5x^3] = 5(-1)^3 = -5$$

Typically, substitution works if it yields a real number. There is a big "if"—that is, a condition—that must be met for the numerical result of substitution to be the limit. The "if" relies on the idea of *continuity*, which we will not define until the next section. We will state the condition below hoping the reader will for now remember that the condition exists and will understand the condition later (after Lecture 3).

If f is continuous on the interval (a, b) , then direct substitution will find the value of $\lim_{x \rightarrow c} f(x)$ for any c such that $a < c < b$. In other words, if f is continuous on the interval (a, b) , then $\lim_{x \rightarrow c} f(x) = f(c)$ for any c such that $a < c < b$.

According to the condition above, direct substitution will work for polynomial functions, exponential functions, sine waves, and cosine waves. Moreover, for values where they are defined, direct substitution will also work for logarithmic functions, rational functions, and the remaining four trigonometric functions.

Consider the function $f(x) = 1/x$, a rational function defined for all non-zero values of x . To evaluate $\lim_{x \rightarrow 0.1} f(x)$, we can apply direct substitution: $\lim_{x \rightarrow 0.1} [1/x] = 1/(0.1) = 10$. Simple enough, but the astute reader should be worried about $\lim_{x \rightarrow 0} f(x)$ because the function is not defined at zero and direct substitution will not yield a real number. To evaluate $\lim_{x \rightarrow 0} f(x)$, we must recognize $x = 0$ as a vertical asymptote approached by $f(x)$ as shown below.



The graph shows that $\lim_{x \rightarrow 0^-} f(x) = -\infty$, $\lim_{x \rightarrow 0^+} f(x) = \infty$, and $\lim_{x \rightarrow 0} f(x)$ does not exist.

Limit Properties

In the previous example, when direct substitution did not yield the limit, the limit did not exist. There are times when the limit does exist, but direct substitution will not find it. At such times, several properties of limits may apply.

First, limits are linear operators, that is, the limit of a sum is the sum of the limits and the limit of a scalar multiple of a function is the scalar multiple of the limit of the function as stated below.

Suppose that c is a constant and that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} [f(x)] \pm \lim_{x \rightarrow a} [g(x)]$$

and

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} [f(x)].$$

Second, the limit of a product equals the product of the limits.

Suppose that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} [f(x)] \cdot \lim_{x \rightarrow a} [g(x)].$$

Third, the limit of a quotient equals the quotient of the limits provided that the limit of the divisor does not equal zero.

Suppose that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} [f(x)]}{\lim_{x \rightarrow a} [g(x)]} \text{ if } \lim_{x \rightarrow a} [g(x)] \neq 0.$$

From the second property, we have a fourth property stated below.

Suppose n is a positive integer and $\lim_{x \rightarrow a} f(x)$ exists, then

$$\lim_{x \rightarrow a} [(f(x))^n] = \left[\lim_{x \rightarrow a} f(x) \right]^n$$

and

$$\lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \text{ provided that } \lim_{x \rightarrow a} f(x) > 0 \text{ if } n \text{ is even.}$$

Lecture 2

Finally, we have the most useful property.

If $f(x) = g(x)$ when $x \neq a$ and $\lim_{x \rightarrow a} f(x)$ exists, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x).$$

The first four properties are fairly intuitive and easy to apply. The fifth property deserves exposition. Consider the limit below.

$$\lim_{x \rightarrow -2} \frac{x+2}{x^2-4}$$

Direct substitution does not yield a number (recall that division by zero is undefined). Note, however, that $\frac{x+2}{x^2-4} = \frac{1}{x-2}$ for all values of x except negative two. Accordingly, we may apply our fifth property, to find the limit.

$$\lim_{x \rightarrow -2} \frac{x+2}{x^2-4} = \lim_{x \rightarrow -2} \frac{\cancel{x+2}}{(x-2)\cancel{(x+2)}} = \lim_{x \rightarrow -2} \frac{1}{x-2} = \frac{1}{(-2)-2} = \frac{1}{-4} = -\frac{1}{4}$$

Here, we reduced. Other times, we may rationalize as below.

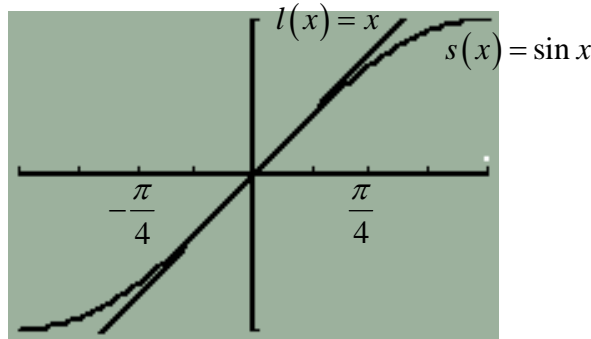
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x^2+25}-5}{x^2} &= \lim_{x \rightarrow 0} \frac{\sqrt{x^2+25}-5}{x^2} \cdot \frac{\sqrt{x^2+25}+5}{\sqrt{x^2+25}+5} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2+25})^2 - (5)^2}{x^2(\sqrt{x^2+25}+5)} \\ &= \lim_{x \rightarrow 0} \frac{x^2+25-25}{x^2(\sqrt{x^2+25}+5)} \\ &= \lim_{x \rightarrow 0} \frac{\cancel{x^2}}{\cancel{x^2}(\sqrt{x^2+25}+5)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2+25}+5} = \frac{1}{\sqrt{(0)^2+25}+5} = \frac{1}{\sqrt{25}+5} = \frac{1}{10} \end{aligned}$$

Before a final example, we will introduce a special limit:

Lecture 2

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1. *$$

This limit means that as x approaches zero (from both sides), the sine-wave function $s(x) = \sin x$ behaves like the linear function $l(x) = x$. Examining the graphs of s and l makes it less surprising that the ratio $(\sin x)/x$ approaches one as x -values approach zero.



Now, that we are convinced that $\lim_{x \rightarrow 0} [(\sin x)/x] = 1$, let's evaluate the limit below.

$$\lim_{x \rightarrow 0} \frac{\sin cx}{x}$$

Applying our favorite property, we can multiply the function by the fraction c/c to get:

$$\lim_{x \rightarrow 0} \frac{\sin cx}{x} = \lim_{x \rightarrow 0} \frac{c \sin cx}{cx}$$

Considering that $c \cdot x$ will approach zero when x approaches zero, we can write:

$$\lim_{x \rightarrow 0} \frac{\sin cx}{x} = \lim_{cx \rightarrow 0} \frac{c \sin cx}{cx}$$

Allowing $z = cx$, we can substitute z for $c \cdot x$ then evaluate using the property that limits are linear operators, we find:

$$\lim_{x \rightarrow 0} \frac{\sin cx}{x} = \lim_{cx \rightarrow 0} \frac{c \sin cx}{cx} = \lim_{z \rightarrow 0} \frac{c \sin z}{z} = c \cdot \left(\lim_{z \rightarrow 0} \frac{\sin z}{z} \right) = c \cdot 1 = c.$$

* Direct substitution for $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ yields $\frac{0}{0}$, which is undefined and does not equal one. It is only a coincidence

that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

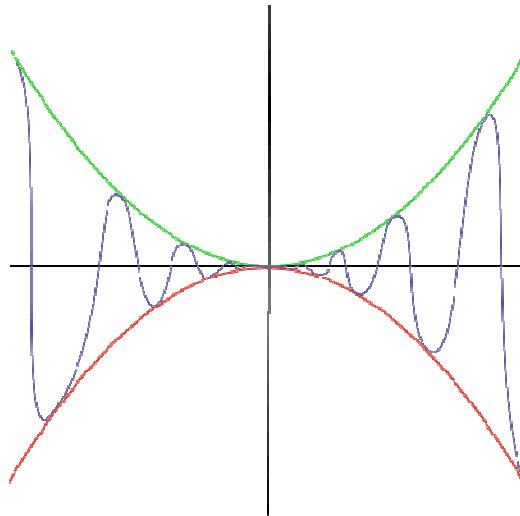
The Squeeze Theorem

Now we turn to a theorem called the Squeeze or Sandwich or Pinching theorem that is useful only in special cases. Informally, the *Squeeze Theorem* states that if two functions approach the same limit L at a , and if $y(x)$ is "squeezed" between those functions for all values near a (except possibly a itself), then $y(x)$ also approaches that limit L at a . The box below states the theorem formally.

The Squeeze Theorem: If \exists some interval I containing a where $f(x) \leq y(x) \leq c(x)$ for all values along I except possibly at a and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c(x) = L$, then

$$\lim_{x \rightarrow a} y(x) = L.$$

A famous example of a "squeezed" function is $y(x) = x^2 \sin(1/x)$, which is "sandwiched" or "pinched" or "squeezed" between $f(x) = -x^2$ and $c(x) = x^2$ as shown below.



We know that $\lim_{x \rightarrow 0} (x^2) = 0$ and $\lim_{x \rightarrow 0} (-x^2) = -\lim_{x \rightarrow 0} (x^2)$. Since $-x^2 \leq x^2 \sin(1/x) \leq x^2$ for all values of x except 0, then by the Squeeze Theorem, $\lim_{x \rightarrow 0} [x^2 \sin(1/x)] = 0$.

Practice Problems

1st ed. problem set: 2.3 #1–17 odd

2nd ed. problem set: 2.3 #1–19 odd

3rd ed. problem set: 2.3 #1–21 odd

Possible Exam Problems

#1 What does it mean to say, "Limits are linear operators"?

$$\text{Answer: } \lim_{x \rightarrow a} [f(x) \pm g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \pm \left[\lim_{x \rightarrow a} g(x) \right] \text{ and } \lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \left[\lim_{x \rightarrow a} f(x) \right].$$

#2 Evaluate $\lim_{x \rightarrow 11} \frac{22x^2 - 220x - 242}{x^2 - 121}$.

$$\text{Answer: } \lim_{x \rightarrow 11} \frac{22x^2 - 220x - 242}{x^2 - 121} = 22 \cdot \lim_{x \rightarrow 11} \frac{x^2 - 10x - 11}{x^2 - 121} = 22 \cdot \lim_{x \rightarrow 11} \frac{x+1}{x+11} = 22 \left(\frac{12}{22} \right) = 12$$

#3 Evaluate $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{1 - x}$.

$$\text{Answer: } \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{1 - x} = \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{1 - x} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \rightarrow 1} \frac{x - 1}{(1 - x)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1} \frac{-1}{\sqrt{x} + 1} = -\frac{1}{2}$$

#4 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 2\theta}{\theta}$.

Answer: 2

#5 Use the Squeeze Theorem to compute $\lim_{x \rightarrow \infty} \left[\frac{2 - \cos x}{x + 3} \right]$.

Answer: Note that $-1 \leq -\cos(x) \leq 1$. Adding two, we obtain $1 \leq 2 - \cos(x) \leq 3$. Since x approaches infinity, we know $x + 3 > 0$; hence, we can divide by $x + 3$ to obtain,

$$\frac{1}{x+3} \leq \frac{2 - \cos(x)}{x+3} \leq \frac{3}{x+3}.$$

Since $\lim_{x \rightarrow \infty} \frac{1}{x+3} = \lim_{x \rightarrow \infty} \frac{3}{x+3} = 0$, then $\lim_{x \rightarrow \infty} \frac{2 - \cos(x)}{x+3} = 0$ by the Squeeze Theorem.

Example Exercise 1

Suppose $f(x) = x^3 - 3$.

Evaluate: $\lim_{x \rightarrow 0} f(x)$.

Since $f(x)$ is continuous (a concept defined strictly in the next section, but for now simply means that it has no interruptions), we can use direct substitution to evaluate the limit.

$$\lim_{x \rightarrow 0} f(x) = 0^3 - 3 = -3$$

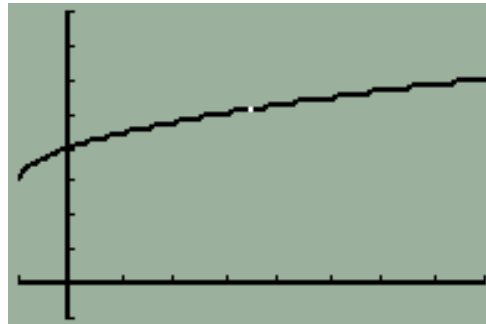
Example Exercise 2

Suppose $r(x) = \sqrt{x+1} + 3$.

Evaluate: $\lim_{x \rightarrow -1^+} r(x)$.

As x approaches -1 from the right, the function approaches 3. This is evident, both from the table and from the graph of $r(x)$.

X	Y ₁
0	4
-.5	3.7071
-.9	3.3162
-.99	3.1
-.999	3.0316
-.9999	3.01
X =	



Example Exercise 3

$$\text{Evaluate: } \lim_{x \rightarrow 0} \left[\frac{(x+3)^3 - 27}{x} \right].$$

Simplifying the argument of the limit, we obtain the following.

$$\begin{aligned} \frac{(x+3)^3 - 27}{x} &= \frac{(x+3)(x+3)(x+3) - 27}{x} \\ \frac{(x+3)^3 - 27}{x} &= \frac{(x^3 + 9x^2 + 27x + 27) - 27}{x} \\ \frac{(x+3)^3 - 27}{x} &= \frac{x^3 + 9x^2 + 27x}{x} \\ \frac{(x+3)^3 - 27}{x} &= x^2 + 9x + 27, \quad x \neq 0 \end{aligned}$$

Our favorite property of limits is the property that $\lim_{x \rightarrow a} f = \lim_{x \rightarrow a} g$ if $f = g$ everywhere but possibly when $x = a$. Hence,

$$\lim_{x \rightarrow 0} \left[\frac{(x+3)^3 - 27}{x} \right] = \lim_{x \rightarrow 0} (x^2 + 9x + 27) = 0^2 + 9 \cdot 0 + 27 = 27.$$

Example Exercise 4

$$\text{Suppose } \lim_{x \rightarrow 2} [f(x)] = 4 \text{ and } \lim_{x \rightarrow 2} [g(x)] = 3. \text{ Evaluate } \lim_{x \rightarrow 2} [2f(x) + 3g(x)].$$

Limits are linear operators; therefore, the limit of a sum equals the sum of the limits and the limit of a scaled function equals the scalar times the limit of the un-scaled function as below.

$$\lim_{x \rightarrow 2} [2f(x) + 3g(x)] = 2 \lim_{x \rightarrow 2} [f(x)] + 3 \lim_{x \rightarrow 2} [g(x)]$$

Substituting given values, we obtain the desired value.

$$\begin{aligned} \lim_{x \rightarrow 2} [2f(x) + 3g(x)] &= 2 \cdot 4 + 3 \cdot 3 \\ \lim_{x \rightarrow 2} [2f(x) + 3g(x)] &= 17 \end{aligned}$$

Example Exercise 5

$$\text{Evaluate: } \lim_{x \rightarrow 9} \left[\frac{9-x}{\sqrt{x}-3} \right].$$

Simplifying the argument of the limit, we obtain the following.

$$\begin{aligned} \frac{9-x}{\sqrt{x}-3} &= \frac{9-x}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \\ &= \frac{(9-x)(\sqrt{x}+3)}{x-9} \\ &= \frac{-1(\cancel{x-9})(\sqrt{x}+3)}{\cancel{x-9}} \\ &= -1(\sqrt{x}+3) \end{aligned}$$

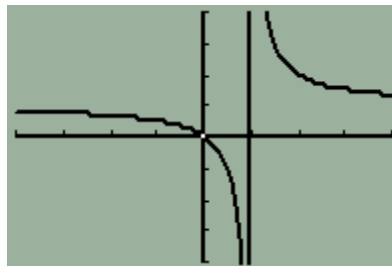
Our favorite property of limits is the property that $\lim_{x \rightarrow a} f = \lim_{x \rightarrow a} g$ if $f = g$ everywhere but possibly when $x = a$. Hence,

$$\lim_{x \rightarrow 9} \left[\frac{9-x}{\sqrt{x}-3} \right] = \lim_{x \rightarrow 9} \left[-1(\sqrt{x}+3) \right] = -1 \left[\lim_{x \rightarrow 9} (\sqrt{x}+3) \right] = -1 \left[\sqrt{9}+3 \right] = -6.$$

Example Exercise 6

$$\text{Evaluate: } \lim_{x \rightarrow 1} \left[\frac{x}{x-1} \right].$$

Graphing the argument of the limit as a function reveals a vertical asymptote at $x = 1$. Hence, the limit does not exist.



$$\lim_{x \rightarrow 1} \left[\frac{x}{x-1} \right] = D.N.E.$$

Example Exercise 7

$$\text{Evaluate: } \lim_{x \rightarrow 0} \left[\frac{x \sin(x)}{|x|} \right].$$

Note that $\frac{x}{|x|} = \pm 1$. Multiplying by $\sin(x)$, we obtain:

$$\frac{x \sin(x)}{|x|} = \pm \sin(x)$$

Hence, we can write $-\sin x \leq \frac{x \sin(x)}{|x|} \leq \sin(x)$. We note $\lim_{x \rightarrow 0}(\sin x) = \lim_{x \rightarrow 0}(-\sin x) = 0$. Thus,

by the Squeeze Theorem, $\lim_{x \rightarrow 0} \left[\frac{x \sin(x)}{|x|} \right] = 0$.

Example Exercise 8

$$\text{Evaluate: } \lim_{x \rightarrow 2} \left[\frac{x^2 - 7x + 10}{x^2 - 4} \right].$$

Simplifying the argument of the limit, we obtain the following.

$$\begin{aligned} \frac{x^2 - 7x + 10}{x^2 - 4} &= \frac{\cancel{(x-2)}(x-5)}{(x+2)\cancel{(x-2)}} \\ \frac{x^2 - 7x + 10}{x^2 - 4} &= \frac{(x-5)}{(x+2)}, \quad x \neq 2 \end{aligned}$$

Our favorite property of limits is the property that $\lim_{x \rightarrow a} f = \lim_{x \rightarrow a} g$ if $f = g$ everywhere but possibly when $x = a$. Hence,

$$\lim_{x \rightarrow 2} \left[\frac{x^2 - 7x + 10}{x^2 - 4} \right] = \lim_{x \rightarrow 2} \left[\frac{(x-5)}{(x+2)} \right] = \frac{2-5}{2+2} = -\frac{3}{4}.$$

Application Exercise

Nobel prize-laureate physicist Hendrik Lorentz described a physical phenomenon called length contraction, which is the decrease in the length of objects that travel at any non-zero velocity relative to an observer.

The Lorentz length contraction formula below expresses the length L of an object as a function of its velocity relative to an observer.

$$L(v) = L_0 \left(1 - \frac{v^2}{c^2} \right)^{\frac{1}{2}}$$

In the formula, L_0 represents the initial length while c represents the speed of light. In other words, L_0 is a constant equal to the length of the object at rest and c is a constant equal to the speed of light.

Use the information provided above to evaluate $\lim_{v \rightarrow c^-} [L(v)]$.