Lecture 19

Optimization

Lecture 17 discussed the extreme values of functions. This lecture will apply the lesson from Lecture 17 to word problems. In this section, it is important to remember we are in Calculus *I* and are dealing *one*-variable functions. Calculus III deals with multi-variable functions, but we are interested for now in one-variable functions. Accordingly, if we need to find the maximum volume of a cylinder given by $V = \pi r^2 h$, then we need to be able to substitute for *r* in terms of *h* or for *h* in terms of *r* so that the volume formula becomes a function in one-variable.

Let's start off easy and find two nonnegative numbers that add to 44 such that the product of the two addends is as large as possible. First, name the variables and any restrictions. We are looking for two numbers, so we will name them *x* and *y* and note that $0 \le x \le 44$ and $0 \le y \le 44$. Second, write the function to be optimized in this case $P = xy$. Third, find a relation between the variables in this case $x + y = 44$ so $y = 44 - x$. Fourth, reduce the function to be optimized to one variable using the relation from step three: $P = x(44 - x)$. Fifth, we find the extreme value of interest using the function from step four using either the first derivative or the second derivative test.

$$P = 44x - x^2$$
$$P' = 44 - 2x$$
$$44 - 2x = 0$$
$$44 = 2x$$
$$22 = x$$
$$P'' = -2 < 0$$

$\therefore x = 22$ corresponds to local maximum by second derivative test

Finally, we use the information from step five to answer the question: The two nonnegative numbers that add to 44 such that their product is as large as possible are 22 and 22 (remember that $y = 44 - x$ and $44 - 22 = 22$). ("Hey!" The reader says, "22 and 22 are not *two* numbers." Yes, 22 and 22 is a pair of numbers; they just are not distinct. The problem did not say that the two numbers must be distinct.)

Let's summarize the steps above then use the steps to solve another stated problem.

1. Name the variables and restrictions.
2. Write the function to be optimized.
3. Find a relation between the variables.
4. Reduce the function from step two to one variable using the relation from step three.
5. Find the extreme values using the first derivative or second derivate test.
6. Answer the question.

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Here's a typical optimization problem: If a closed tin can in the shape of a right-circular cylinder of volume $16\pi\ \text{in}^3$, find the height and radius if the least amount of material is used to manufacture the tin can. According to step one, we name the variables and restrictions. We need to find height and radius: *h* and *r*, both of which must be positive numbers because a can cannot have negative or zero dimensions. The material used is measured in surface area, which is given by $S = 2\pi rh + 2\pi r^2$. Thus, we have the function to be minimized (step two). We can use the given volume to find a relation between the variables (step three):

$$V = \pi r^2 h$$
$$16\pi = \pi r^2 h$$
$$h = \frac{16}{r^2}$$

Substituting $16/r^2$ for *h* into the surface area formula accomplishes step four (reducing the function to be optimized to one variable). Now, we perform the second derivative test:

$$S = 2\pi r\left(\frac{16}{r^2}\right) + 2\pi r^2$$
$$S = 32\pi r^{-1} + 2\pi r^2$$
$$S' = -32\pi r^{-2} + 4\pi r$$

$$4\pi r^{-2}\left(-8 + r^3\right) = 0$$
$$4\pi r^{-2}\left(r^3 - 8\right) = 0$$
$$\frac{4\pi}{r^2}(r-2)\left(r^2 + 2r + 4\right) = 0$$
$$\frac{4\pi}{r^2} \neq 0 \quad r - 2 = 0 \quad r^2 + 2r + 4 \neq 0$$
$$r = 2$$

$$S'' = 64\pi r^{-3} + 4\pi$$
$$S''(2) = \frac{64\pi}{2^3} + 4\pi$$
$$S''(2) = 12\pi > 0$$

$\therefore$ $r = 2$ corresponds to local minimum by the second derivative test

Since $h = 16/r^2$, then when $r = 2$, $h = 4$. Recall that the units given were in inches and state the answer: the least material will be used if the can is 4 inches high and 2 inches in diameter.

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Practice Problems

1st ed. problem set:	Section 4.6	#3, #5, #13
2nd ed. problem set:	Section 4.6	#3–11 odd, #19
3rd ed. problem set:	Section 4.6	#3–13 odd, #19

Possible Exam Problem

#1 A box must be manufactured to have a volume of 288 cubic inches where the base is a rectangle three times its width. What dimensions of the box require the least material?

Answer: 12 inches $\times$ 4 inches $\times$ 6 inches

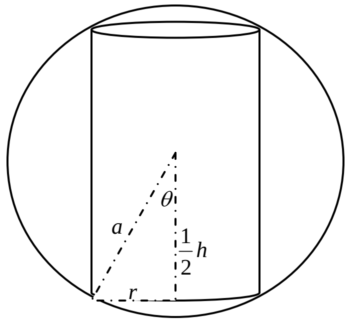
#2 Determine the dimensions of the largest field that can enclosed using 500 feet of fencing material with one side the field bordered by a river so that fencing is not needed on that side.

Answer: 250 feet $\times$ 125 feet

Example Exercise

A right-circular cylinder is to be inscribed in a sphere of constant radius a. What is the ratio of the altitude to the base radius of the cylinder having the largest lateral surface area $(S = 2\pi rh)$?

The function has two variables. We need to relate the two variables. Draw a diagram.



Note the right triangle with the base radius as one leg, one-half the height of the cylinder as the second leg, and the sphere's radius as the hypotenuse. Write equations relating the base radius and the height to the angle between the hypotenuse and the vertical leg.

$$\begin{aligned} \sin\theta &= r/a \\ r &= a\sin\theta \end{aligned} \quad \text{and} \quad \begin{aligned} \cos\theta &= 0.5h/a \\ h &= 2a\cos\theta \end{aligned}$$

Rewrite the function so that it depends on one variable. Remember that the sphere's radius is constant.

$$S = 2\pi\left(a\sin\theta\right)\left(2a\cos\theta\right)$$
$$S = 2\pi a^2\left(2\sin\theta\cos\theta\right)$$
$$S = 2\pi a^2 \sin 2\theta$$

Take the derivative and find the critical numbers.

$$S' = 4\pi a^2 \cos 2\theta$$
$$4\pi a^2 \cos 2\theta = 0$$
$$\cos 2\theta = 0$$
$$\theta = \pi/4$$

Evaluating the second derivative at this critical number yields a negative value as below.

$$S'' = -8\pi a^2 \sin 2\theta$$
$$S''\left(\pi/4\right) = -8\pi a^2$$

Hence, $\theta = \pi/4$ maximizes the function. Substituting into the equations above, we see $r = a/\sqrt{2}$ and $h = 2a/\sqrt{2}$. Therefore, the ratio $h/r = 2$.

Application Exercise

Suppose that a manufacturer requires that the aluminum containers for its product have a capacity of 54 cubic inches and take the shape of a right circular cylinder. Approximate the radius and height of the container that requires the least amount of aluminum.