

Concavity

Lecture 16 looked at the derivative of f in order to find local extrema on f and to make conclusions about the behavior of f . This lecture will discuss how to make conclusions about the shape of f using the second derivative of f . Because it is a gainful exercise, we will first run through process of using imagined or superimposed tangent lines on f , to roughly sketch f' .

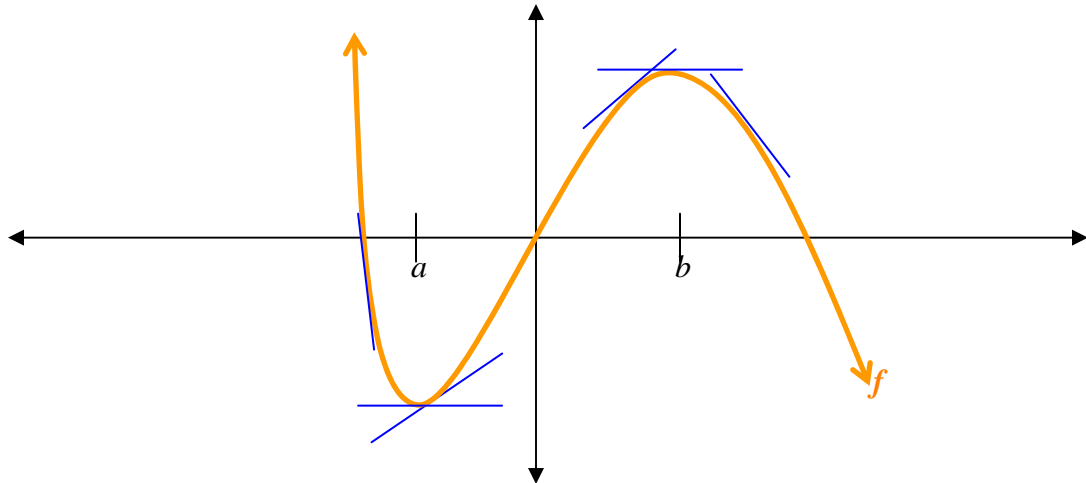


Figure 1

Examining the graph in Figure 1, we see that the lines tangent to the curve at $x = a$ and $x = b$ are horizontal lines with a slope of zero. It follows, then, that a and b are real roots of f' . We also see that the tangent lines on the interval $(-\infty, a)$ have negative slopes as do the lines tangent to f at points on the interval (b, ∞) . From this we gather that $f' < 0$ on $(-\infty, a) \cup (b, \infty)$. Finally, we note that lines tangent to f on the intervals (a, b) all have positive slopes, which means $f' > 0$ over the same interval. These conclusions are summarized by the graph of f' below in Figure 2, but now we will repeat the process in order to envision f'' .

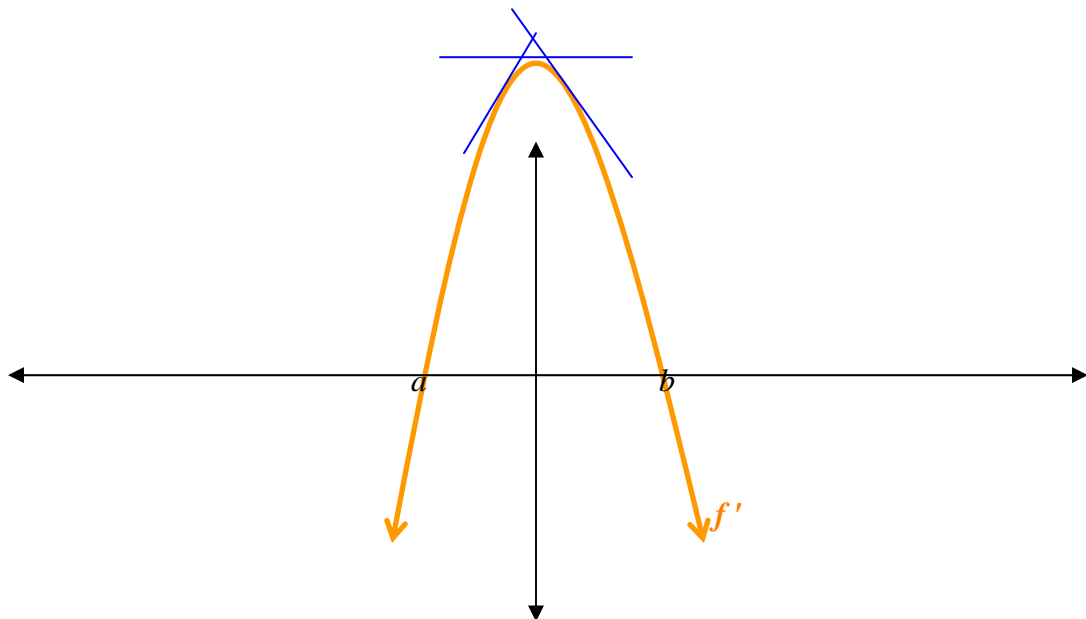


Figure 2

Clearly, the line tangent to f' at $x = 0$, has a slope of zero. Just as clearly, lines tangent to f' when $x < 0$ have positive slopes while those tangent to f' when $x > 0$ have negative slopes. We conclude, therefore, that f'' is positive on the interval $(-\infty, 0)$, negative on the interval $(0, \infty)$, and equal to zero at zero as summarized by the graph below.

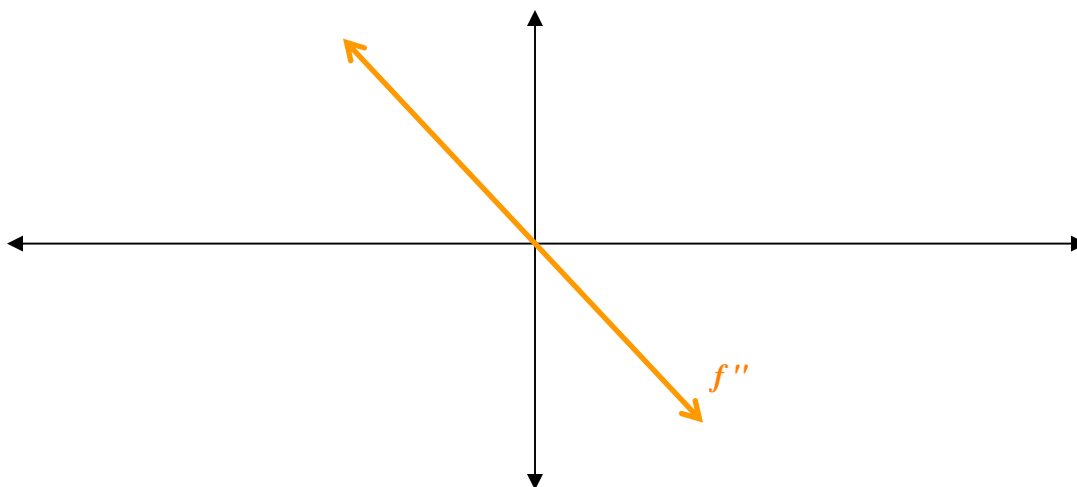


Figure 3

Before we compare the graph of f in Figure 1 to the graph of f'' in Figure 3, we will state a definition.

The shape of the graph of a function is said to be *concave up* on some interval I if f' is increasing on I . The shape of the graph of a function is said to be *concave down* on I if f' is a decreasing on I .

Now, we are ready to compare the graph of f in Figure 1 to the graph of f'' in Figure 3. Doing so, we notice that where $f'' > 0$, f is concave up because f' is increasing and where $f'' < 0$, f is concave down because f' is decreasing. We can generalize this observation with the following theorem.

The Concavity of f Theorem: If $f''(x) > 0$ on an interval, then f is concave up on that interval, and if $f''(x) < 0$ on an interval, then f is concave down on that interval.

Further comparison of the graph of f in Figure 1 to the graph of f'' in Figure 3 reveals that the root of f'' corresponds to a change in concavity on f . The point where f changes from one type of concavity to another is called an *inflection point*.

Assume f is continuous over the interval $[a, b]$ that contains c such that $a < c < b$. If f changes concavity at the point $(c, f(c))$, then the point $(c, f(c))$ is an *inflection point*.

The Concavity of f Theorem combined with the definition of an inflection point gives us another useful conclusion stated below.

Inflection Theorem: Assume f is a continuous function on some interval I containing only one critical number c of f' . Then $(c, f(c))$ is an inflection point if each of the following conditions is true.

1. $f''(c) = 0$.
2. The interval I contains a and b such that $a < c < b$.
3. $f''(a)$ has the opposite sign of $f''(b)$.

Consider $f(x) = 3x^5 + 5x^3$. Assume we want to find intervals of concavity and inflection points on f . We need the second derivative, which, of course, requires the first derivative.

$$\begin{aligned} f(x) &= 3x^5 + 5x^3 \\ f'(x) &= 15x^4 + 15x^2 \\ f''(x) &= 60x^3 + 30x \end{aligned}$$

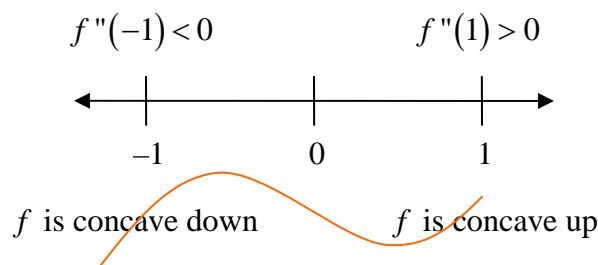
Since $f''(x)$ is a polynomial function continuous over the number line, the only critical numbers of $f'(x)$ are the roots of $f''(x)$.

$$\begin{aligned} 60x^3 + 30x &= 0 \\ 30x(2x^2 + 1) &= 0 \\ x &= 0 \end{aligned}$$

Zero is the only root of $f''(x)$, so we conclude that zero is the only critical number of $f'(x)$.

Incidentally, since $f''(0) = 0$, the point $(0, f(0))$ is a *possible* inflection point on the graph of f .

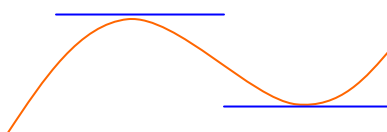
To determine if $(0, f(0))$ is a point of inflection and to find the intervals of concavity, we investigate the sign of f'' on either side of the critical numbers of f' . Because we have Rolle's Theorem and all the roots of f'' , it is only necessary to choose one value for each interval left of and right of every critical number of f' , and any *one* number on each interval will tell us the sign of f'' for every x -value on the interval so that we can make conclusions about the concavity of f on that interval.



Since f'' is negative on the interval $(-\infty, 0)$, we conclude f is concave down over $(-\infty, 0)$. Similarly, since f'' is positive on the interval $(0, \infty)$, we conclude f is concave up over $(0, \infty)$. Moreover, we can conclude by the Inflection Theorem that $(0, f(0))$ is a point of inflection because f'' changes sign over an interval containing zero and since $f''(0) = 0$. The process of testing the sign of f'' on either side of the critical numbers of f' to find intervals of concavity and inflection points is called the *Concavity Test*.

Second Derivative Test

The alert reader may have noticed that the sketch of the concavity of $f(x) = 3x^5 + 5x^3$ implies a local maximum appears within the interval where f is concave down. The sketch also implies a local minimum appears where f is concave up.



These implications provide an alternative method for finding local extreme, namely the *Second Derivative Test*.

Second Derivative Test: If $x = c$ is a critical point of f where $f'(c) = 0$, then

1. If $f''(c) > 0$, f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, f has a local maximum at $(c, f(c))$.
3. If $f''(c) = 0$, this test is inconclusive.
4. If $f''(c)$ does not exist, this test fails.

If the test is inconclusive at c , then f could have a local minimum or a local maximum or neither at c . If the test fails at c , then the First Derivative Test must be employed (see Lecture 16).

Practice Problems in *Calculus: Concepts and Contexts* by James Stewart

- 1st ed. problem set: Section 2.10 #1 all parts, #11 all parts, #15–17 odd
 Section 4.3 #3–5 odd, #7–11 part *c* only, #13–25 odd (all parts)
- 2nd ed. problem set: Section 2.10 #1 all parts, #11 all parts, #15–17 odd, #23
 Section 4.3 #3–5 odd, #7–13 part *c* only, #15, #17–29 odd (all parts)
- 3rd ed. problem set: Section 2.9 #1 all parts, #11 all parts, #15–17 odd, #23
 Section 4.3 #3–5 odd, #7–13 part *c* only, #15, #19–35 odd (all parts)

Practice Problems in *Calculus: Early Transcendentals* by Briggs and Cochran

- 1st ed. problem set: 4.2 #47-61 odd

Possible Exam Problem

#1 Show that $f(x) = \ln(x)$ has no inflection points.

Answer: $f''(x) = -\frac{1}{x^2}$, which is negative for all values of x . By the concavity theorem, f is always concave down. Since f never changes concavity, it has no points of inflection.

#2 Sketch the graph of $h(x) = e^{-x^2}$ labeling all intercepts, extrema, inflection, asymptotes, and intervals of behavior and concavity.

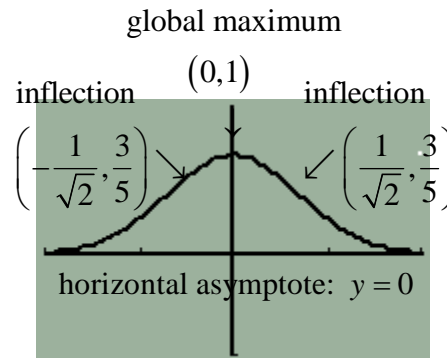
Answer:

increasing $(-\infty, 0)$

decreasing $(0, \infty)$

concave up $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

concave down $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



Example Exercise 1

Find the point of inflection to the left of the origin on the curve described by $y = x \ln|x|$.

Note that the function is equivalent to the piece-wise function below by the definition of the absolute value.

$$y = x \ln|x| = \begin{cases} x \ln(-x) & \text{if } x < 0 \\ x \ln(x) & \text{if } x \geq 0 \end{cases}$$

Since we are interested in the curve to the left of the origin, we consider the case where $x < 0$.

$$\begin{aligned} y &= x \ln(-x) \\ y' &= x \cdot \frac{d}{dx}[\ln(-x)] + \ln(-x) \cdot \frac{d}{dx}[x] \\ y' &= x \cdot \frac{-1}{-x} + \ln(-x) \cdot 1 \\ y' &= 1 + \ln(-x) \end{aligned}$$

Now we set this derivative equal to zero.

$$\begin{aligned} 1 + \ln(-x) &= 0 \\ \ln(-x) &= -1 \\ -x &= e^{-1} \\ x &= -\frac{1}{e} \end{aligned}$$

Remember that this is the derivative of the function for negative x -values. Note that y' is negative to the left of $-1/e$, and y' is positive to the right of $-1/e$. Hence, the function is concave down along $(-\infty, -1/e)$ then concave up along $(-1/e, 0)$. By the Inflection Theorem a point of inflection occurs on the curve when $x = -1/e$. Substituting $-1/e$ for x into the function, we find the point of inflection.

$$\begin{aligned} y &= -\frac{1}{e} \cdot \ln\left|-\frac{1}{e}\right| \\ y &= -\frac{1}{e} \cdot -1 \\ y &= 1/e \end{aligned}$$

The point of inflection left of the origin is $\left(-\frac{1}{e}, \frac{1}{e}\right)$.

Example Exercise 2

Describe the graph of $f(x) = x^3 + 3x^2 - 4$. Use calculus to identify intervals of behavior and concavity. Use calculus to identify extrema and inflection points. Use calculus to demonstrate end-behavior, that is, the behavior of the curve as x approaches $\pm\infty$.

Examine the end-behavior of the function by determining the limit of the function as x approaches $\pm\infty$.

$$\lim_{x \rightarrow \infty} (x^3 + 3x^2 - 4) = (\infty)^3 + 3(\infty)^2 - 4 = \infty$$

$$\lim_{x \rightarrow -\infty} (x^3 + 3x^2 - 4) = (-\infty)^3 + 3(-\infty)^2 - 4 = -\infty$$

From this analysis, we see that the graph rises on the far right and falls on the far left.

Determine the derivative to apply the Behavior Theorem.

$$f(x) = x^3 + 3x^2 - 4$$

$$f'(x) = 3x^2 + 6x$$

The roots of f' represent critical numbers of f .

$$3x(x+2) = 0$$

$$\text{CN: } -2, 0$$

Note that $f'(-3) > 0$, $f'(-1) < 0$, $f'(1) > 0$. Hence, f increases along $(-\infty, -2) \cup (0, \infty)$ and decreases along $(-2, 0)$, which means $f(-2) = 0$ represents a local maximum while $f(0) = -4$ represents a local minimum.

Determine the second derivative to apply the Concavity Theorem & Inflection Theorem.

$$f'(x) = 3x^2 + 6x$$

$$f''(x) = 6x + 6$$

The roots of f'' represent critical numbers of f' .

$$6(x+1) = 0$$

$$\text{CN: } -1$$

Note that $f''(-2) < 0$ and $f''(0) > 0$. Hence, f is concave down along $(-\infty, -1)$ and concave up along $(-1, \infty)$, which indicates that $f(-1) = -2$ is a point of inflection on the graph.

Application Exercise

Curves described by $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ occur in probability and statistics.

The constant μ represents the *mean* of a population and the positive constant σ represents the *standard deviation* of the population. Suppose for a particular population $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2\pi}}$. What are the inflection points of the curve?