Global Extrema

The previous lecture looked at the derivative of f in order to find local extrema on f. In this lecture, we will make a distinction between local extrema and global (or absolute) extrema. In layman's terms, global extrema correspond to points that contain the greatest or least value of f over an interval [a,b]. Consider the graph of f in Figure 1 below.

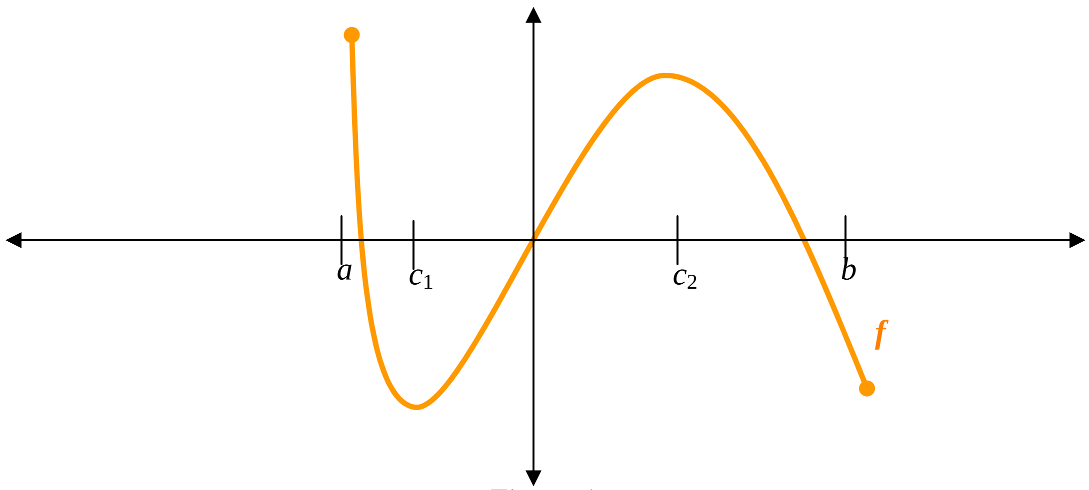

Figure 1

Over the interval [a,b], the point $(c_1, f(c_1))$ is a local minimum while the point $(c_2, f(c_2))$ is a local maximum. Of the two points, only $f(c_1)$ is a global extremum because $f(c_1)$ is the least value of f on [a,b]. The local maximum at $f(c_2)$ is *not* a global maximum because $f(a)$, not $f(c_2)$, is the greatest value of f on [a,b].

Interestingly, continuity at point a is a requirement for $f(a)$ to be a local extremum, but continuity at a is not a requirement for $f(a)$ to be a global extremum. Consequently, it is worth mentioning that in Figure 1 $f(b)$ is not considered either a global nor a local minimum on the interval [a,b]. The value $f(b)$ is not the least value of f on [a,b], so it is not a global minimum. Similarly, $f(b)$ is not a local minimum because f is not continuous at b.

Formally, we define *global extrema* (or *absolute extrema*) as below.

> A function f has a *global maximum* (also called an *absolute maximum*) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the maximum value of f on D. Similarly, f has a *global minimum* (also called an *absolute minimum*) at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D. The maximum and minimum values of f are called the *extreme values* of f.

The Extreme Value Theorem follows from the definition above.

Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Consider the graphs of g and h in Figure 2. Using the Extreme Value Theorem, we can say that g is guaranteed to contain extreme values while h is not.

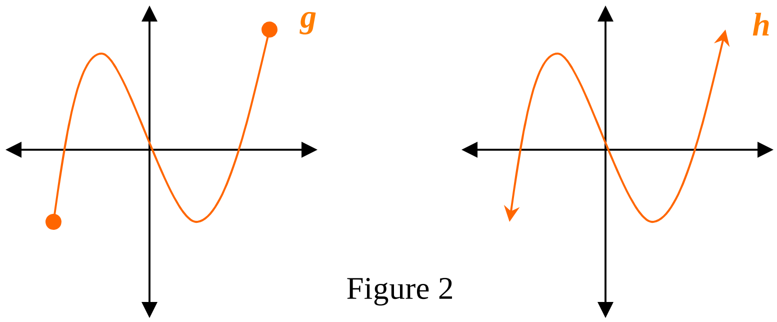

Figure 2

Practice Problems

1st ed. problem set:	Section 4.2	#7–9 odd, #15–23 odd, #37–45 odd
2nd ed. problem set:	Section 4.2	#7–9 odd, #15–21 odd, #35–43 odd
3rd ed. problem set:	Section 4.2	#7–9 odd, #15–21 odd, #37–47 odd

Possible Exam Problems

#1 Consider $F(x)=\sqrt{x}$ with a domain of $[0,\infty)$. Does *F* have a global minimum? If so what is it? Under what conditions can *F* be guaranteed to have a global maximum.

Answer: *F* has a global minimum of zero at $x=0$. Restricting the domain so that some number *c* serves as an upper bound on the domain of *F*, i.e., restricting the domain to $[0,c]$ guarantees that *F* has a global maximum by the Extreme Value Theorem.

#2 Find the absolute maximum of $q(x)=\frac{4x-4}{x^2}$.

Answer: $q(2)=1$

Example Exercise

Find the vertex for the parabola $p(x)=2x^2+6x+1$.

To apply the Behavior Theorem, find p'.

$$p(x)=2x^2+6x+1$$
$$p'(x)=4x+6$$

Set p' equal to zero, to find the critical numbers of *p*.

$$4x+6=0$$
$$4x=-6$$
$$x=-6/3$$
$$x=-1.5$$

Note that $p'(-2)<0$ and $p'(0)>0$. By the Behavior Theorem, the function $p(x)$ decreases along $(-\infty,-1.5)$ then increases along $(-1.5,\infty)$. Hence, the vertex, a global minimum, occurs at $p(-1.5)=-3.5$.

Application Exercise

The decomposition of organic waste dumped into a pond reduces oxygen levels in the pond. Suppose the function below gives the oxygen content of the pond as a percent of its natural (pre-contamination) level *t* days after contamination.

$$P(t) = 100\left(\frac{t^2 + 10t + 100}{t^2 + 20t + 100}\right)$$

What is the absolute lowest level of oxygen (expressed as a percent of pre-contamination levels) that the pond suffers after contamination? How many days does it take to reach this minimum level?