

Curve Sketching

Consider the function f whose graph appears in Figure 1, and think about the slopes of the tangent lines to the curve.

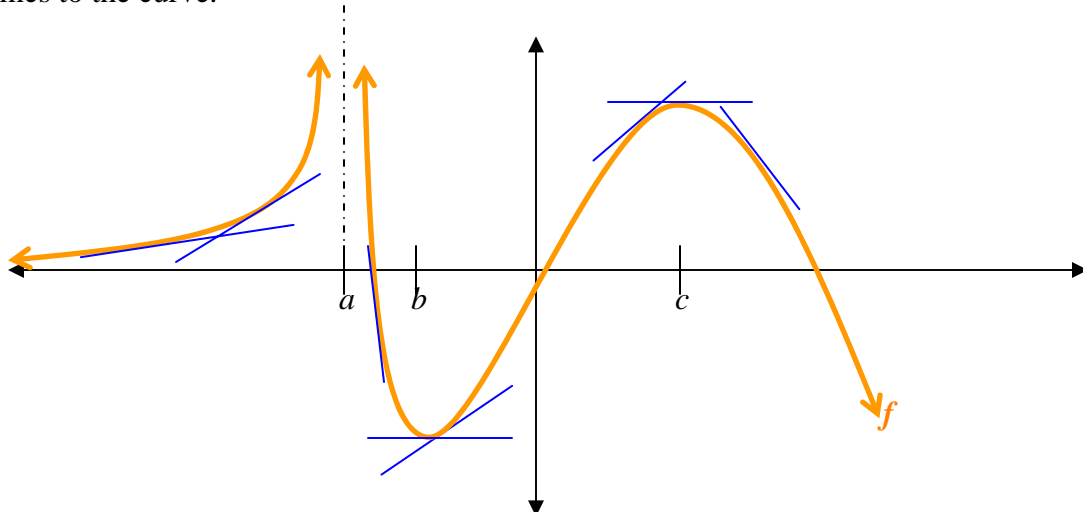


Figure 1

It should be fairly obvious that the lines tangent to the curve at $x = b$ and $x = c$ are horizontal lines with a slope of zero. It follows, then, that b and c are real roots of f' . It should be equally obvious that the tangent lines on the interval (a, b) have negative slopes as do the lines tangent to f at points on the interval (c, ∞) . From this we gather that $f' < 0$ on $(a, b) \cup (c, \infty)$. Finally, it is clear that lines tangent to f on the intervals $(-\infty, a) \cup (b, c)$ all have positive slopes, which means $f' > 0$ over the same intervals. All this information is summarized by the graph of f' below in Figure 2.

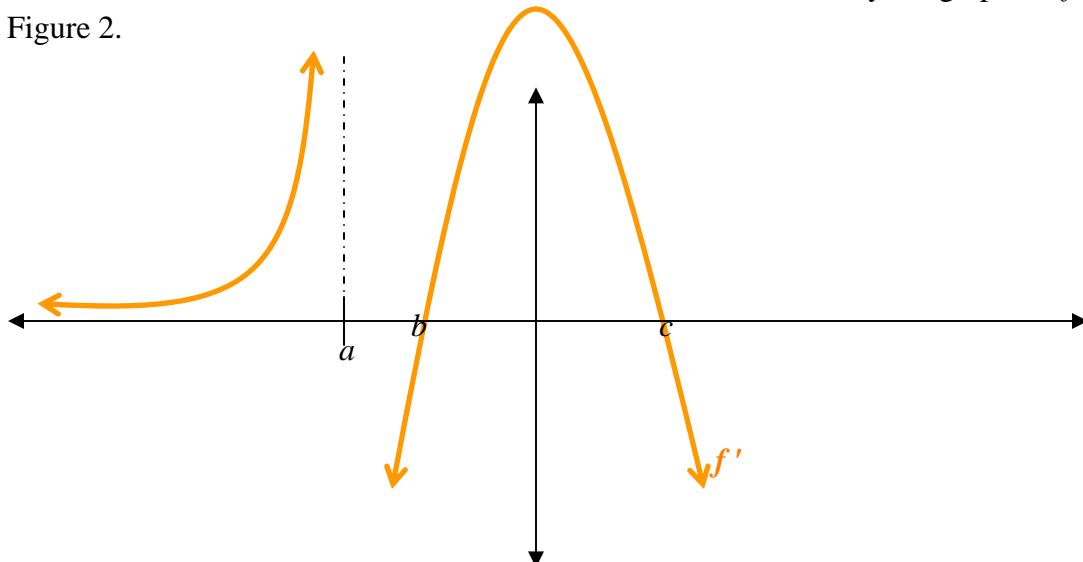
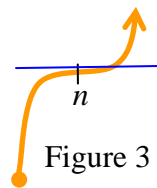


Figure 2

Notice that where $f' > 0$, f is increasing and where $f' < 0$, f is decreasing. We can generalize this statement, and we will.

The Behavior Theorem: If $f'(x) > 0$ on an interval, then f is increasing on that interval, and if $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Returning to our discussion of Figure 1 and Figure 2. Notice that where $f' = 0$ or is undefined there occurs a change in behavior in f from increasing to decreasing or vice versa. This statement is not always true. Imagine that f was defined in such a way that the graph of f in Figure 1 was reflected over the x -axis over the interval $(-\infty, a)$, but the remainder of the graph remained as is. In such a case, the discontinuity at a would not correspond to a change from one behavior to another since f would be decreasing on both sides of a . Another exception would occur on a curve like g shown below at $x = n$ where g' would equal zero but would not correspond to a change in behavior in g since g increases on either side of n .



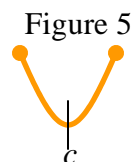
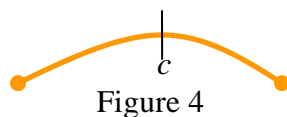
Nevertheless, points where $f' = 0$ or is undefined correspond to locations where f could change behavior. Moreover, if these points occur in the domain of f , they could correspond to a "peak" or "turn-around" in f , for this reason we will define the points along the x -axis where $f' = 0$ or is undefined as *critical numbers* of f if they occur in the domain of f .

A critical number of function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Remember that critical numbers are x -values where f might have a "peak" or "turn-around." We want to make use of this fact. To do so, we first need a definition.

If there exists an interval of values (a, b) containing c with $a < c < b$ or $b < c < a$ over which f is continuous and such that $f(c) \geq f(x)$ or $f(c) \leq f(x)$ for all x on the interval, then $f(c)$ is a *local extremum*, also called a *relative extremum*. If $f(c) \geq f(x)$ for all x on $[a, b]$, then $f(c)$ is a *local (or relative) maximum*. If $f(c) \leq f(x)$ for all x on $[a, b]$, then $f(c)$ is a *local (or relative) minimum*.

This definition gives us a term for the "turn-around points" on f . If the turn-around occurs at a "peak," it is a local maximum (see Figure 4). If the turn-around occurs at a "valley," it is a local minimum (see Figure 5).



Lecture 16

With local extrema defined, we can proceed to *Fermat's Theorem*.

Fermat's Theorem: If f has a local maximum or local minimum at c , then c is a critical number of f .

One interesting aspect of Fermat's Theorem is that the converse is not necessarily true. As we stressed in the previous paragraph, if c is a critical number of f , then f may or may not have a local maximum or local minimum at c . Nevertheless, Fermat's Theorem is a powerful tool for sketching curves because it allows us to think in the converse then investigate.

For example, let's consider $f(x) = x^5 + 2x^4 + x^3$. Suppose we want to sketch the curve of this polynomial. We are interested in its intercepts, behavior, and extrema. Determining the intercepts is uninteresting, so we will just list them as $(0, -1)$ and $(0, 0)$. To determine behavior and to find extrema, we will find f' as below.

$$f(x) = x^5 + 2x^4 + x^3$$

$$f'(x) = 5x^4 + 8x^3 + 3x^2$$

Next, we will find the critical numbers of f —that is, the x -values such that f' does not exist or is equal to zero. Since f' is a polynomial and defined for all x -values, we conclude that the roots of f' are the only critical numbers of f .

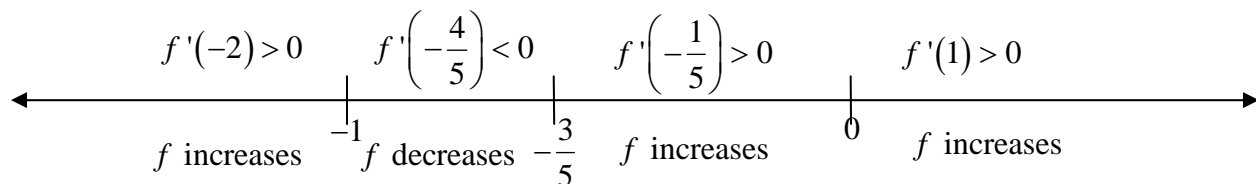
$$5x^4 + 8x^3 + 3x^2 = 0$$

$$x^2(5x^2 + 8x + 3) = 0$$

$$xx(5x + 3)(x + 1) = 0$$

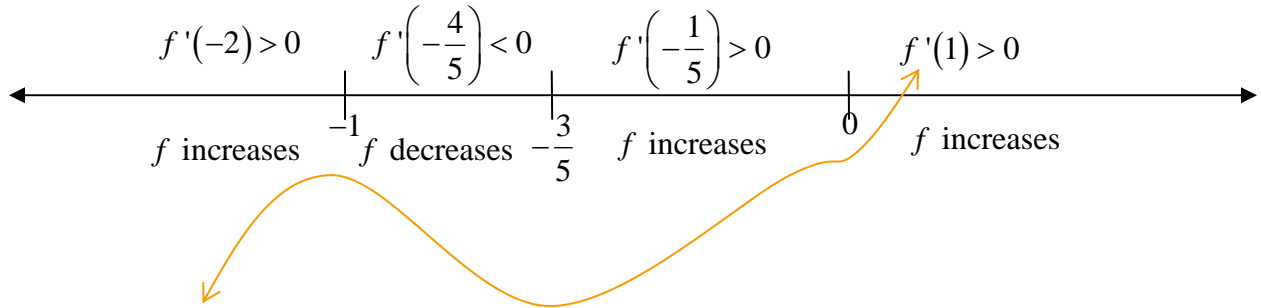
$$x = 0, \quad x = -\frac{3}{5}, \quad x = -1$$

Thinking of the converse of Fermat's Theorem, we suspect that these three critical numbers correspond to extrema on f , so we investigate by plotting the three numbers on a number line and evaluating f' for values left and right of each one, making conclusions about f as we go. Because we have Rolle's Theorem and all the roots of f' , it is only necessary to choose one value for each interval left of, right of, or between critical numbers, and any *one* number on each interval will tell us the sign of f' for every x -value on the interval.



Lecture 16

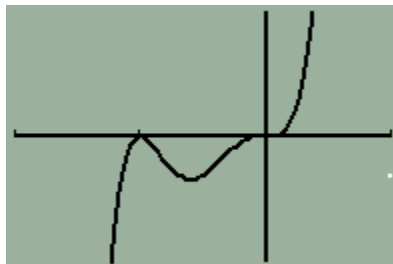
Applying the same logic that supports Rolle's Theorem to f' , we know that f' is positive on the intervals $(-\infty, -1) \cup (-3/5, 0) \cup (0, \infty)$. Correspondingly, by the Behavior Theorem, we conclude that f increases on the same interval $(-\infty, -1) \cup (-3/5, 0) \cup (0, \infty)$. Similarly, we know f' is negative on the interval $(-1, -3/5)$ and f decreases along $(-1, -3/5)$. Note, that f does not change its behavior at the critical number zero, which shows the converse of Fermat's Theorem is not true for all critical numbers. At -1 and at $-3/5$, however, the function does change behavior, which indicates that $f(-1)$ and $f(-3/5)$ are local extrema. Selecting an arbitrary interval $[a, b]$ such that -1 greater than one endpoint but less than the other and comparing $f(-1)$ to the value of the points near it, we can determine if $f(-1)$ is a minimum or maximum. We do the same for $-3/5$. Or, we can employ our number line diagram. If we superimpose a sketch of general behavior that matches our conclusions, it becomes apparent that $f(-1)$ must be a local maximum while $f(-3/5)$ must be a local minimum.



The process of testing the sign of f' on either side of the critical numbers of f is called the *First Derivative Test*.

To identify the actual extrema-values, we simply evaluate $f(-1)$ and $f(-3/5)$. Since $f(-3/5) = -0.03456$, we know -0.03456 is a local minimum that occurs when $x = -3/5$.

Since $f(-1) = 0$, we know zero is a local maximum that occurs when $x = -1$. Plotting the local extrema and intercepts and heeding the intervals of increasing and decreasing behavior, we can sketch the curve with fair accuracy.



Practice Problems

- 1st ed. problem set: Section 2.10 #1, #11 parts a and b only, #19, #20
 Section 4.2 #25–35 odd
 Section 4.3 #7–11 odd parts a & b only, #13–19 odd parts a & b only
- 3rd ed. problem set: Section 2.9 #1, #11 parts a & b only, #21, #22, #23 part a only
 Section 4.2 #23—35 odd
 Section 4.3 #7–13 odd parts a & b only, #19–27 odd parts a & b only

Possible Exam Problems

#1 State the Behavior Theorem.

Answer: *The Behavior Theorem:* If $f'(x) > 0$ on an interval, then f is increasing on that interval, and if $f'(x) < 0$ on an interval, then f is decreasing on that interval.

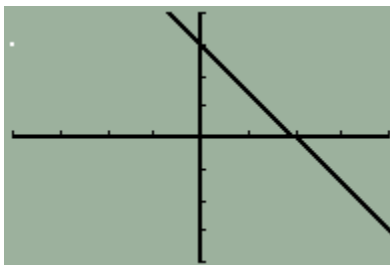
#2 State the definition of a critical number of f .

Answer: A *critical number* of function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

#3 Given $g(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$. Find the critical numbers of g .

Answer: 0 and 1.5

#4 Given the graph of H' below, what conclusions can we make regarding H . Assume the scales of the x and y axes is 1 unit per tick mark.



Answer: $H(2)$ is a local maximum. H increases along $(-\infty, 2)$ & decreases along $(2, \infty)$.

Application Exercise

Let x represent the parts per million of carbon monoxide in the ambient atmosphere (ranging from 500 to 800). Suppose the function $T(x)$ gives the approximate duration in hours of human exposure before victims will fall into a carbon monoxide induced coma.

$$T(x) = 0.0002x^2 - 0.316x + 127.9$$

Consider the domain of $T(x)$, which is $500 \leq x \leq 800$. Over what interval of the domain does the time before a person succumbs to carbon monoxide poisoning decrease?