Applications of Continuity

Consider a plane flying at an altitude of 11,000 feet at 9:00 AM. If at 9:15 AM the plane lands at JFK airport, can we assume the plane was at some time between 9:00 AM and 9:15 AM flying at an altitude of 4,750 feet? Well, yes, quite obviously so. Our plane exemplifies an application of continuity called the *Intermediate Value Theorem*.

> The *Intermediate Value Theorem*: If f is a continuous function on the closed interval $[a,b]$ and M is any number between $f(a)$ and $f(b)$, then there exists at least one number c in $[a,b]$ such that $f(c) = M$.

The Intermediate Value Theorem states that if f is continuous over the interval $[a,b]$ and M is some number between $f(a)$ and $f(b)$, then there must be a number c in the interval $[a,b]$ such that $f(c) = M$ as illustrated in Figure 1 below.

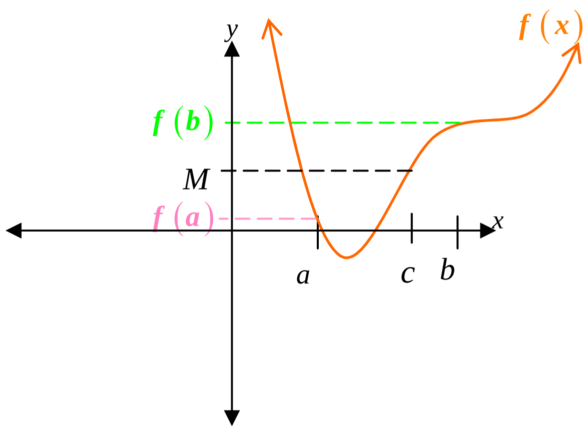


Figure 1

In the example of the airplane, the function f gives the altitude of the plane at time t. Since the plane has an altitude at all times between 9:00 AM and 9:15 AM, the function is continuous on the interval [9:00 AM, 9:15 AM]. Using the designations from the formal statement of the theorem, $a = 9:00$ AM , $b = 9:15$ AM , $f(a) = 11{,}000$ feet , $f(b) = 0$ feet , and M can be any altitude between 11,000 feet and zero feet such as 4,750 feet. Using the *Intermediate Value Theorem*, we know that there is some time between 9:00 and 9:15 AM where the plane's altitude was 4,750 feet.

Suppose we need to estimate the roots of a polynomial function $p(x)$. We know $p(x)$ is continuous because it is a polynomial function. If we know $p(1) = 2.3$ and $p(2) = -1.7$, then we know that there is some number c such that $1 < c < 2$ and $p(c) = 0$. In other words, we know there is a root between 1 and 2 by the Intermediate Value Theorem.

Let's return to our airplane. If the plane takes off from JFK airport at 9:30 AM and lands at DFW airport at 11:30 AM, can we assume that there exists some maximum altitude that the plane reaches during the two-hour interval of its flight? Yes, and doing so is an application of *Rolle's Theorem*.

> *Rolle's Theorem*: If f is a continuous function differentiable on the interval [a,b] and if $f(a)=0$ and $f(b)=0$, then there exists at least one point c in the interval [a,b], where $f'(c)=0$.

Rolle's Theorem states that if $f(x)$ is continuous and $f(a)=0$ and $f(b)=0$, then there is at least one point on $f(x)$ where $f'(x)=0$, which is equivalent to saying that somewhere on the interval [a,b], $f(x)$ has a tangent line that is parallel to the x-axis. Figure 2 illustrates Rolle's Theorem.

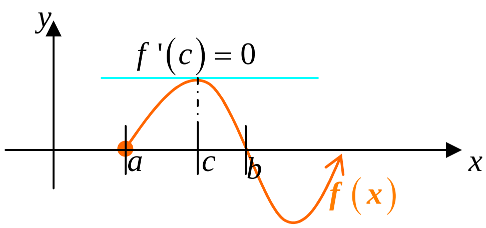


Figure 2

In the example of the airplane, a and b refer to the plane's take off and landing times respectively while c refers to the time (or times) between take off and landing when the plane reached its greatest altitude. Notice that in general, the theorem is stated in such a way that f could have a some maximum or some minimum value (or even some constant value). In the case of the plane, we know it begins its flight by increasing altitude, so we know there is some maximum. What if the plane lands at 11:30 AM and takes off again at 11:45 AM? Do we know there is some maximum (or minimum) altitude? This is a case where the function remains constant. The maximum and minimum altitude for the 11:30 to 11:45 AM interval is zero. Since constant functions are horizontal lines, the tangents to a constant are also horizontal and Rolle's Theorem applies.

Above we stated that Rolle's Theorem concludes that " if $f(x)$ is continuous and $f(a)=0$ and $f(b)=0$, then $f(x)$ has a tangent line that is parallel to the x-axis." We could have said Rolle's Theorem concludes that "if $f(x)$ is continuous and $f(a)=0$ and $f(b)=0$, then $f(x)$ has a tangent line that is parallel to the line containing the endpoints of the graph on the interval [a,b]." If we had, and if we also did not assume that $f(a)=0=f(b)$ but that $f(a)$ and $f(b)$ could be any value, then the conclusion to Rolle's Theorem gives us the *Mean Value Theorem*.

> *Mean Value Theorem*: If f is a continuous function differentiable on the interval [a,b] , then there exists at least one point c in the interval [a,b], where $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Note that $\frac{f(b)-f(a)}{b-a}$ is nothing more than the slope formula applied to the endpoints of *f* on the interval [*a*,*b*]. If $f'(c)=\frac{f(b)-f(a)}{b-a}$, then the line tangent to *f* at *c* is parallel to the line going through the endpoints.

Applications of Limits

Consider a population of animals living in a habitat. If the habitat is free of predators the population might grow very rapidly, but all habitats have natural barriers to growth like food supply. Once the population reaches a certain point, food becomes scarce and the population reaches a limit. This limit is called the carrying capacity of the habitat. Often populations of this type can be modeled with functions of the form $g(x)=\frac{c}{1+ae^{-kx}}$, where *k* is a positive constant and *c* is a positive number called the carrying capacity or upper *asymptotic value*. An asymptotic value is some number that the function may approach as *x* grows very large (or very small). Graphically, the function approaches a horizontal line called a *horizontal asymptote*. Formally, the horizontal asymptote of *g* can be defined with limits:

$$y=\lim_{x\to\infty}\frac{c}{1+ae^{-kx}}=\frac{c}{1+ae^{-k\infty}}=\frac{c}{1+a\left(e^{-\infty}\right)^k}=\frac{c}{1+a\left(\frac{1}{e^{\infty}}\right)^k}=\frac{c}{1+a\cdot 0^k}=\frac{c}{1+a\cdot 0}=\frac{c}{1}=c\,.$$

In general, we define horizontal asymptotes as below.

> A *horizontal asymptote* is a line approached by a function *f* as *x* approaches infinity or negative infinity with the equation
>
> $$y=\lim_{x\to\pm\infty}f(x).$$

For example, consider $f(x)=\frac{12x-1}{2-3x}$. To see if *f* has an asymptote, we evaluate $\lim_{x\to\infty}\frac{12x-1}{2-3x}$. Direct substitution yields $\frac{\infty}{\infty}$, which is said to be indeterminant. Accordingly, we multiply the argument of the limit by a judicious form of 1 and simplify.*

* In this situation, a judicious 1 will be $\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$ where *n* equals the degree of the denominator.

$$\lim_{x\to\infty}\frac{12x-1}{2-3x}=\lim_{x\to\infty}\frac{12x-1}{2-3x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{12x}{x}-\frac{1}{x}}{\frac{2}{x}-\frac{3x}{x}}=\lim_{x\to\infty}\frac{12-\frac{1}{x}}{\frac{2}{x}-3}$$

Now, direct substitution finds the limit as shown below.

$$\lim_{x\to\infty}\frac{12-\frac{1}{x}}{\frac{2}{x}-3}=\frac{12-\frac{1}{\infty}}{\frac{2}{\infty}-3}=\frac{12-0}{0-3}=\frac{12}{-3}=-4$$

Accordingly, we know that the function approaches a horizontal asymptote described by the equation $y=-4$ when x-values grow very large. Performing a similar calculation for $x\to-\infty$ shows that the function approaches the same asymptote as x grows very small.

In the preceding paragraph, we glossed over the term "indeterminant." There are certain expressions that are designated as indeterminants. These expressions are listed below.

Indeterminants include indeterminant ratios:

1. $\frac{0}{0}$, 2. $\frac{\infty}{\infty}$;

the indeterminant product:

3. $0\cdot\infty$;

the indeterminant difference:

4. $\infty-\infty$;

and the indeterminant powers:

5. 0^0, 6. ∞^0, 7. 1^∞.

Notice that ∞ and $-\infty$ are not listed as indeterminants. If $\lim_{x\to a} f(x)=\pm\infty$, then the limit does not exist because the expression grows ever larger (or ever smaller). If the limit involves infinity (or negative infinity), then the function does not approach an asymptote as x grows very large (or as x becomes very small). In other words, if $\lim_{x\to\infty} f(x)=\pm\infty$, then $f(x)$ does not have a horizontal asymptote to the right, and if $\lim_{x\to-\infty} f(x)=\pm\infty$, then $f(x)$ does not have a horizontal asymptote to the left. If both $\lim_{x\to\infty} f(x)=\pm\infty$ and $\lim_{x\to-\infty} f(x)=\pm\infty$, then $f(x)$ does not have a horizontal asymptote at all.

For some functions, it is helpful to recognize the horizontal asymptote. Consider functions of the form $y = b^x$ where $b > 1$. Using the limit as x approaches negative infinity, we see that $y = 0$ is a horizontal asymptote for exponential functions with base greater than 1.

$$\lim_{x\to-\infty} b^x = b^{-\infty} = \frac{1}{b^{\infty}} = \frac{1}{\infty} = 0$$

Let's consider the case for functions of the form $y = b^x$ where $0 < b < 1$. If $0 < b < 1$, then there exists some number $c > 1$ such that $b = 1/c$. Therefore, $b^x = (1/c)^x$. Accordingly, we can evaluate the limit as below.

$$\lim_{x\to\infty} b^x = \lim_{x\to\infty}\left(\frac{1}{c^x}\right) = \frac{1}{c^{\infty}} = \frac{1}{\infty} = 0$$

It follows that a vertical translation will shift a horizontal asymptote vertically, so we accept the following theorem.

> If $f(x) = b^x + c$ where c is any real number and b is any positive number not equal to one, then $y = c$ is a horizontal asymptote approached by f as x approaches negative infinity (when $b > 1$) or as x approaches infinity (when $0 < b < 1$).

Of course, the function $y = e^x$ is a case where $b > 1$ and $c = 0$ while $y = e^{-x}$ is a case where $0 < b < 1$ and $c = 0$.

Some functions have two horizontal asymptotes. One such case is detailed in the box below.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

Besides horizontal asymptotes, some functions have *vertical asymptotes*. A vertical asymptote is a vertical line that the graph will approach as x values approach some number a.

> The vertical line $x = a$ is a *vertical asymptote* of the function f if
>
> $$\lim_{x\to a^+} f(x) = \infty,\ \lim_{x\to a^+} f(x) = -\infty,\ \lim_{x\to a^-} f(x) = \infty,\ \textit{or}\ \lim_{x\to a^-} f(x) = -\infty.$$

The conjunction "or" is significant, and it implies that if any one of the four statements above are true, then $x = a$ is a vertical asymptote of the function.

Consider $f(x) = \dfrac{12x-1}{2-3x}$, which is undefined when $x = 2/3$. Frequently, rational functions have vertical asymptotes at their point of discontinuity. To see if $x = 2/3$ is a vertical asymptote of $f(x)$, we must consider the one-sided limits as x approaches $2/3$:

$$\lim_{x\to\frac{2}{3}^-}\frac{12x-1}{2-3x} \text{ and } \lim_{x\to\frac{2}{3}^+}\frac{12x-1}{2-3x}.$$

To intuitively determine these two limits, we can generate a table of values demonstrating the behavior of $y = f(x)$ as *x* approaching $2/3$.

X	Y1	
.55	16	
.6	31	
.65	136	
.66667	ERROR	
.67	-704	
.7	-74	
1	-11	

X=.666666666667

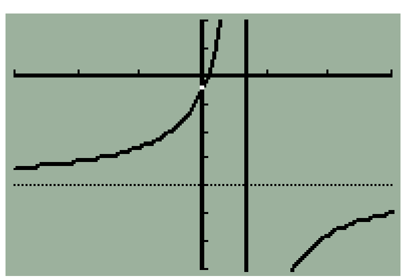
From the left, the function jumps from 16 to 31 to 136 as *x* approaches $2/3$, so we conclude that

$$\lim_{x\to\frac{2}{3}^-}\frac{12x-1}{2-3x}=\infty.$$

This is enough to conclude that $x = 2/3$ is a vertical asymptote approached by $f(x)$. We also see, however, that from the right, the function falls from –11 to –74 to –704 as *x* approaches $2/3$, which indicates

$$\lim_{x\to\frac{2}{3}^+}\frac{12x-1}{2-3x}=-\infty.$$

This provides more evidence that $x = 2/3$ is a vertical asymptote of $f(x)$ as demonstrated by its graph. Note, too, the horizontal asymptote identified earlier as $y = -4$.



Practice Problems

1st ed. problem set: Section 2.4 #31–37 odd & Section 2.5 #5–9 odd, #15–31 odd
2nd ed. problem set: Section 2.4 #35–39 odd & Section 2.5 #5–9 odd, #13–31 odd
3rd ed. problem set: Section 2.4 #35–39 odd & Section 2.5 #5–11 odd, #15–35 odd

Possible Exam Problems

#1 Given $f\left(x\right)=x^3+3x^2-5x-15$. Prove that *f* has a positive root.

Answer: By $\mathbb{R}-\text{Continuous Theorem}$ *f* is continuous over [2,3]. Since $f\left(2\right)=-5$ and $f\left(3\right)=24$ and $-5<0<24$, there exists by the Intermediate Value Theorem some number *c* in the interval [2,3] such that $f\left(c\right)=0$.

#2 Given $Q\left(x\right)=\dfrac{ax^2+bx+c}{d-\frac{1}{2}x^2}$ where *a*, *b*, *c*, and *d* are real numbers, prove that $y=-2a$ is a horizontal asymptote approached by $Q\left(x\right)$.

Answer: By definition: $y=\lim\limits_{x\to\infty}\dfrac{ax^2+bx+c}{d-\frac{1}{2}x^2}$.

Multiplying by one, we can write:

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{d-\frac{1}{2}x^2}=\lim_{x\to\infty}\frac{ax^2+bx+c}{d-\frac{1}{2}x^2}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{\frac{d}{x^2}-\frac{1}{2}}.$$

By direct substitution, we have:

$$\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{\frac{d}{x^2}-\frac{1}{2}}=\frac{a+\frac{b}{\infty}+\frac{c}{\infty^2}}{\frac{d}{\infty^2}-\frac{1}{2}}.$$

A number divided by infinity approaches zero, so we have:

$$\frac{a+\frac{b}{\infty}+\frac{c}{\infty^2}}{\frac{d}{\infty^2}-\frac{1}{2}}=\frac{a}{-\frac{1}{2}}=-2a\ \therefore\ y=-2a$$ is a horizontal asymptote of *Q*.

Example Exercise 1

Consider the function $f(x) = x^3 + 3x - 4$. Use calculus to demonstrate end-behavior, that is, the behavior of the curve as x approaches $\pm\infty$.

Examine the end-behavior of the function by determining the limit of the function as x approaches $\pm\infty$.

$$\lim_{x\to\infty}\left(x^3+3x-4\right)=(\infty)^3+3(\infty)-4=\infty$$
$$\lim_{x\to-\infty}\left(x^3+3x-4\right)=(-\infty)^3+3(-\infty)-4=-\infty$$

From this analysis, we see that the graph rises on the far right and falls on the far left.

Example Exercise 2

Consider the function $Q(x) = \dfrac{5-6x}{2x-3}$. Show that $y=-3$ is a horizontal asymptote of $Q(x)$.

Examine the end-behavior of the function by determining the limit of the function as x approaches $+\infty$ or as x approaches $-\infty$.

$$\lim_{x\to\infty}Q(x)=\lim_{x\to\infty}\left[\frac{5-6x}{2x-3}\right]=\lim_{x\to\infty}\left[\frac{\frac{5}{x}-6}{2-\frac{3}{x}}\right]=\frac{\frac{5}{\infty}-6}{2-\frac{3}{\infty}}=\frac{0-6}{2-0}=\frac{-6}{2}=-3$$

Since Q approaches -3 as x approaches $+\infty$, then $y=-3$ is a horizontal asymptote of Q by definition.

Example Exercise 3

Show that $f(x) = x^5 + 5x - 3$ has a positive root.

The function $f(x) = x^5 + 5x - 3$ is a polynomial and continuous for all real numbers by the $\mathbb{R}$-Continuous Theorem. Moreover, we see that $f(0) = -3$ and $f(1) = 3$. By the *Intermediate Value Theorem*, the function has a root between zero and three. All numbers between zero and three are positive; thus, f has a positive root.

Example Exercise 4

Consider the function $y = \ln(x-1)$. Show that $x = 1$ is a vertical asymptote of $Q(x)$.

Examine the behavior of the function as *x* approaches 1. Since the function is only defined for values greater than 1, find $\lim_{x \to 1^+}\left[\ln(x-1)\right]$, the limit of the function as *x* approaches 1 from the right.

X	Y1
2	0
1.1	-2.303
1.05	-2.996
1.01	-4.605
1.001	-6.908
1	-25.33

X=1.00000000001

Using a table, we see that the function grows smaller as *x* approaches 1 from the right. Hence, $\lim_{x \to 1^+}\left[\ln(x-1)\right] = -\infty$, which means $x = 1$ is a vertical asymptote of the function by definition.

Example Exercise 5

Show that $f(x) = -x^2 + 1$ has a horizontal tangent.

Consider the roots of the function.

$$-x^2 + 1 = 0$$
$$-1\left(x^2 - 1\right) = 0$$
$$-1(x+1)(x-1) = 0$$

Note that f is a continuous differentiable function because it is a polynomial function. Moreover, it has roots at -1 and 1. By *Rolle's Theorem*, there exists at least one value *c* on the interval $(-1,1)$ such that $f'(c) = 0$.

Application Exercise

The decomposition of organic waste dumped into a pond reduces oxygen levels in the pond. Ecologists hope that natural processes will restore the pond's pre-waste oxygen levels given enough time.

Suppose the function below gives the oxygen content of the pond as a percent of its natural (pre-contamination) level *t* days after contamination.

$$P(t) = 100\left(\frac{t^2 + 10t + 100}{t^2 + 20t + 100}\right)$$

What does $\lim_{t\to\infty}\left[P(t)\right]$ represent mathematically and what does $\lim_{t\to\infty}\left[P(t)\right]$ represent to ecologists?