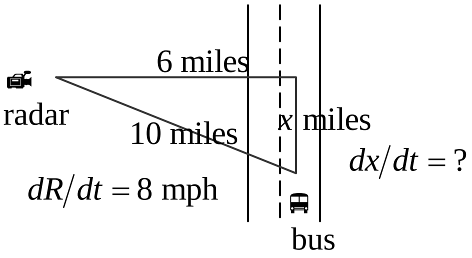
Related Rates

A *related rates* problem involves two or more quantities that change over time. The rates of change are related by some equation; hence, the phrase related rates. For instance, imagine a radar positioned six miles due west of a road running north-south. The radar is tracking the movement of a bus headed due south. The bus is currently ten miles from the radar station, moving away from the station at a rate of eight miles per hour. A related rates problem might ask, "What is the actual speed of the bus?" To solve this problem (and others like it), we will use the six-step process outlined below.

1. Read the problem.
2. Draw a diagram.
3. Identify information tied to a *certain instance* in time.
4. Write an equation relating two variables at *any* time.
5. Differentiate implicitly with respect to time.
6. Substitute the information from step three and solve.

First, we read the problem carefully. Upon first reading the problem from the first paragraph, we may surmise that the speed is given as 8 mph, but that is the rate of change of the bus's distance from the radar unit at the point in time when the bus is ten miles from the radar, not the actual speed of the bus.

Second, we represent the problem with a diagram. For this problem, we draw a road running north south with a radar station due west six miles. Hopefully, a triangle will become apparent. We label the known and unknown quantities and rates. Here, we assign R to represent the distance of the bus from the radar, and x to be the length of the third side in the apparent triangle. The length of the third leg implies the distance the bus travels along a straight path, so that dx/dt represents the bus's speed.



Third, identify the information tied to a certain instance in time. Since the radar station is *always* six miles from the road, the six miles is only relevant in the sense that it gives us x via a simple application of the Pythagorean Theorem. The important information is boxed.

$$R = 10 \text{ miles}$$
$$\frac{dr}{dt} = 8 \text{ mph}$$
$$x = 8 \text{ miles}$$

Fourth, we find an equation that gives the general relationship between the variables. In our case, we again use the Pythagorean Theorem to write, $R^2 = 36 + x^2$. Notice that we did not substitute ten for *R*. That's because we want an equation that relates the two *variables* at any time.

Fifth, differentiate the general equation implicitly with respect to time. The resulting equation should contain at least two derivatives as in our example below.

$$2R\frac{dr}{dt} = 2x\frac{dx}{dt}$$

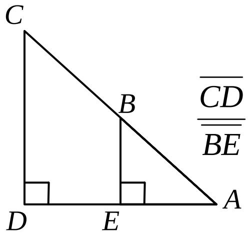
Guess what happens next.

Yes, with the sixth step we substitute the information identified in step three into the differentiated equation and solve. Note that this step must come after the implicit differentiation.

$$\begin{aligned}
&2R\frac{dr}{dt} = 2x\frac{dx}{dt}\\
&2\left(10\text{ miles}\right)\left(\frac{8\text{ miles}}{\text{hour}}\right) = 2\left(8\text{ miles}\right)\frac{dx}{dt}\\
&\frac{160\text{ miles}^2}{\text{hour}} = 16\text{ miles}\frac{dx}{dt}\\
&\frac{10\text{ miles}}{\text{hour}} = \frac{dx}{dt}
\end{aligned}$$

The bus is moving at a speed of ten miles per hour.

If one step must be identified as the most crucial step, it would probably be step four, writing an equation that relates the variables. Commonly, problems involving related rates rely on the Pythagorean Theorem or on the fact that the ratios of side lengths of corresponding sides in two similar triangles are proportional as demonstrated in the diagram below.

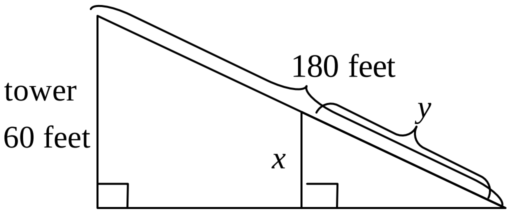


$$\frac{\overline{CD}}{\overline{BE}} = \frac{\overline{DA}}{\overline{EA}} = \frac{\overline{CA}}{\overline{BA}}$$

If the diagram of the problem creates two triangles, then using the proportional ratios of side lengths of two similar triangles is probably required, especially if the two triangles share an angle.

This next problem demonstrates how similar triangles might be used. Consider, for example, a man moving along a 180 foot zip-line like the one pictured below.

The zip-line is suspended from a 60 foot tower. If the man is moving along the zip line at a rate of 15 feet per second, how fast is the man's height above the ground decreasing? Assuming, we have read the problem, the next step is to draw a diagram. If we treat the man as a point on the zip-line, the tower and the vertical height of the man suspended from the zip-line form the heights of two similar right triangles that share the angle formed by the zip-line and the ground. Remember that we desire two variables. Identify the variables in the diagram.



$$\frac{dy}{dt} = -15 \text{ ft/sec}$$

$$\frac{dx}{dt} = ?$$

The next step requires that we gather the information particular to a given time, but we wait to use this information until after differentiating implicitly with respect to time. In this particular problem, no information relevant to some certain instance is given. Instead, the problem gives us the velocity of the man, which is constant (and negative because he is descending). Nevertheless, we can pick some arbitrary instant and say the remaining distance for the man to travel along the zip-line is *y* feet as in our diagram. This makes the man's velocity our rate of change at a particular instance of time.

$$y = y$$

$$\frac{dy}{dt} = -15 \text{ ft per sec}$$

Before we can differentiate, we need an equation relating the two variables at any time. Recall that the ratios of side lengths of corresponding sides in two similar triangles are proportional.

$$\frac{x}{60} = \frac{y}{180}$$

Now, we differentiate implicitly with respect to time.

$$\frac{1}{60}\frac{dx}{dt} = \frac{1}{180}\frac{dy}{dt}$$

Finally, we substitute and solve.

$$\frac{1}{60}\frac{dx}{dt} = \frac{1}{180}\left(\frac{-15 \text{ ft}}{\text{sec}}\right)$$

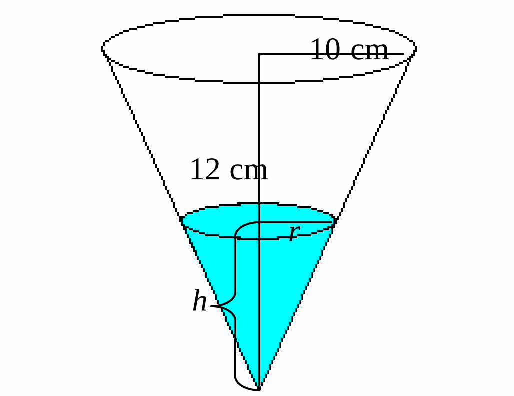
$$\frac{dx}{dt} = 60 \cdot \frac{-1 \text{ ft}}{12 \text{ sec}}$$

$$\frac{dx}{dt} = -5 \text{ ft/sec}$$

The man's height is decreasing by five feet per second.

Other problems my involve volumes or areas. In such problems, volume or area formulas act as the equation relating the two variables. The next example relies on the volume of a cone.

Consider a melted snow cone held upright but leaking through a hole in the narrow-ended bottom. If the cone is twelve centimeters high and ten centimeters in radius at the wide end and the melted ice is draining at a rate of 32 cubic centimeters per second, at what rate is the depth of the fluid going down when the depth is six centimeters? First, we read the problem. Second, we draw a diagram. Hopefully, two similar triangles will be apparent.



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{10}{12} = \frac{r}{h}$$

$$\frac{dV}{dt} = -32 \text{ ft/sec}$$

$$\frac{dh}{dt} = ? \text{ when } h = 6 \text{ cm}$$

Third, we record the information particular to the given time.

$$\boxed{\begin{gathered} \frac{dV}{dt} = -32 \ \text{cm}^3/\text{sec} \\ \\ h = 6 \text{ cm} \end{gathered}}$$

Fourth, we write an equation relating the two variables, height and volume. Since the volume formula has a third variable, which is undesirable, we substitute for *r*. Recall that $\frac{10}{12} = \frac{r}{h}$, so $r = \frac{5}{6}h$, which we substitute into the volume formula.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{5}{6}h\right)^2 h$$

$$V = \frac{25}{108}\pi h^3$$

Fifth, we differentiate implicitly with respect to time.

$$V = \frac{25}{108}\pi h^3$$

$$\frac{dV}{dt} = \frac{25}{36}\pi h^2 \frac{dh}{dt}$$

Lastly, we substitute our particular information and solve for dh/dt.

$$\frac{-32\text{ cm}^3}{\text{sec}} = \frac{25}{36}\pi\left(6\text{ cm}\right)^2 \frac{dh}{dt}$$

$$\frac{-32\text{ cm}^3}{\text{sec}} = 25\pi\text{cm}^2 \frac{dh}{dt}$$

$$\frac{-32\text{ cm}^3}{\text{sec}} \frac{1}{25\pi\text{ cm}^2} = \frac{dh}{dt}$$

$$\frac{-0.407\text{ cm}}{\text{sec}} \approx \frac{dh}{dt}$$

The height is decreasing by about 0.4 centimeters per second.

Lecture 14

Practice Problems

1st ed. problem set: Section 4.1 #1–21 odd, #24
2nd ed. problem set: Section 4.1 #1–25 odd, #30
3rd ed. problem set: Section 4.1 #1–25 odd, #34

Possible Exam Problem

#1 There is a five-foot ladder leaning against a wall. The bottom of the ladder slides away from the wall at a rate of two feet per second in such a way that the top of the ladder stays in contact with the wall. How fast is the top of the ladder moving down the wall when the bottom of the ladder is three feet from the wall?

Answer: The ladder is moving downward 1.5 ft per second.

#2 A 5-foot girls walks toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamp) moving? HINT: A diagram should reveal similar triangles.

Answer: If y represents the distance in feet between the tip of the shadow and the base of the lamppost, then $dy/dt = -8 \text{ ft/sec}$. Hence, the tip of the shadow is moving 8 ft per second toward the lamppost.

#3 Air pumped at the rate of two cubic inches per second inflates a spherical balloon. How fast is the diameter of the balloon increasing when the radius is one-half inch?

Answer: The diameter is increasing at the rate of approximately 1.27 inches per second at the specified radius.

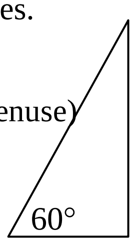
#4 Dockworkers pull a boat to the dock using a rope that passes through a ring on the bow of the boat and a ring on the edge of the dock. The dock is eight feet higher than the ring at the bow of the boat. If the workers pull the rope at the rate of three feet per second, how fast is the boat approaching the dock when the length of the rope between the dock and the boat is ten feet?

Answer: The boat approaches the dock at the rate of 5 feet per second at the specified rope-length.

Example Exercise 1

> An airplane ascends at a speed of 400 kilometers per hour along a line making a 60° angle with the ground. How fast is the altitude changing?

Draw the plane's ascent. Assign variables.



$h =$ length of plane's flight path (hypotenuse)

$y =$ altitude of plane (leg_2)

$x =$ distance over land from take-off (leg_1)

Note that there is no information tied to a specific time. Write an equation that relates two variables at *any* time. Avoid trying to use the plane's speed (apply rates of change after differentiating implicitly with respect to time).

$$\sin(60^{\circ}) = \frac{y}{h}$$
$$\frac{\sqrt{3}}{2} = \frac{y}{h}$$
$$\frac{\sqrt{3}}{2}h = y$$

Differentiate implicitly with respect to time.

$$\frac{\sqrt{3}}{2}\frac{dh}{dt} = \frac{dy}{dt}$$

Now substitute known values. The rate at which the plane's path (hypotenuse) is changing equals 400 kilometers per hour.

$$\frac{\sqrt{3}}{2}\frac{dh}{dt} = \frac{dy}{dt}$$
$$\frac{\sqrt{3}}{2}400\ \frac{\text{km}}{\text{hr}} = \frac{dy}{dt}$$
$$200\sqrt{3}\ \frac{\text{km}}{\text{hr}} = \frac{dy}{dt}$$
$$\frac{dy}{dt} \approx 346\ \text{km}/\text{hr}$$

The plane's altitude is changing at about 346 kilometers per hour.

Example Exercise 2

At a certain moment, a sample of gas obeying Boyle's law, $pV = k$ (k is a constant, p is pressure, V is volume), occupies a volume of 1,000 cubic inches at a pressure of ten pounds per square inch. If a force is compressing the gas at the rate of 12 cubic inches per minute, find the rate at which the pressure is increasing at the instant when the volume is 600 cubic inches.

The equation relating two variables at *any* time is given. The actual value of the constant is unimportant. Differentiate implicitly with respect to time.

$$\begin{aligned} &pV = k \\ &p \cdot \frac{dV}{dt} + V \cdot \frac{dp}{dt} = 0 \end{aligned}$$

Now substitute known values. Since a force is compressing the volume rather than expanding it, use a negative value for the rate of compression.

$$\begin{aligned} &p \cdot \frac{dV}{dt} + V \cdot \frac{dp}{dt} = 0 \\ &\frac{10\text{ lbs}}{\text{in}^2} \cdot \left(\frac{-12\text{ in}^3}{\text{minute}} \right) + 600\text{ in}^3 \cdot \frac{dp}{dt} = 0 \\ &-\frac{120\text{ lbs} \cdot \text{in}}{\text{minute}} + 600\text{ in}^3 \cdot \frac{dp}{dt} = 0 \\ &600\text{ in}^3 \cdot \frac{dp}{dt} = \frac{120\text{ lbs} \cdot \text{in}}{\text{minute}} \\ &\frac{dp}{dt} = \frac{120\text{ lbs} \cdot \text{in}}{\text{minute}} \cdot \frac{1}{600\text{ in}^3} \\ &\frac{dp}{dt} = \frac{0.2\text{ lbs}/\text{in}^2}{\text{minute}} \end{aligned}$$

Example Exercise 3

Consider a pill-shaped solid formed when two hemispheres attach to opposite ends of a cylinder.

If the volume of the figure remains constant while the radius increases at the rate of $1/(2\pi)$ centimeters per minute, how fast must the height of the cylinder be changing when the radius and cylinder's height are both ten centimeters?

Write an equation relating two variables at *any* time. Let h represent the cylindrical height. Let *r* represent the radius (of the cylinder and the hemispheres on either end).

$$V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

Differentiate implicitly with respect to time. Remember that the volume is constant.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 + \pi r^2 h \\ 0 &= 4\pi r^2 \frac{dr}{dt} + \pi r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \frac{dr}{dt} \end{aligned}$$

Substitute known values at a certain instant in time as well as rates of change.

$$\begin{aligned} &4\pi r^2 \frac{dr}{dt} + \pi r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \frac{dr}{dt} = 0 \\ &4\pi (10\text{ cm})^2 \frac{1}{2\pi}\frac{\text{cm}}{\text{minute}} + \pi (10\text{ cm})^2 \cdot \frac{dh}{dt} + (10\text{ cm}) \cdot 2\pi (10\text{ cm}) \frac{1}{2\pi}\frac{\text{cm}}{\text{minute}} = 0 \end{aligned}$$

Solve for dh/dt.

$$\begin{aligned} &\frac{200\text{ cm}^3}{\text{minute}} + 100\pi\text{cm}^2 \cdot \frac{dh}{dt} + \frac{100\text{ cm}^3}{\text{minute}} = 0 \\ &100\pi\text{ cm}^2 \cdot \frac{dh}{dt} = -\frac{300\text{ cm}^3}{\text{minute}} \\ &\frac{dh}{dt} = -\frac{3}{\pi}\frac{\text{cm}}{\text{minute}} \end{aligned}$$

Hence, the cylinder's height decreases at a rate of approximately 0.955 centimeters per minute.

Application Exercise

Oil from an uncapped well in the ocean is radiating outward in the form of a circular film on the surface of the water. If the radius of the circle is increasing at the rate of 2 meters per minute, how fast is the area of the oil film growing when the radius is 100 meters?