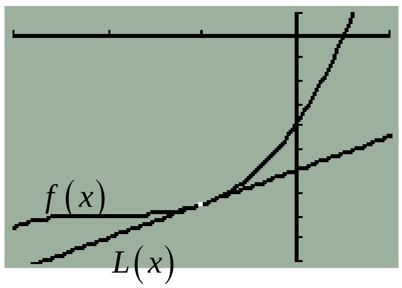
Linearization

Much of what is deemed "higher mathematics" could be described as a series of techniques used to reduce problems to what might be deemed "mundane mathematics." For a case in point, consider the task of evaluating $f(x) = x^3 + 6x^2 + 12x - 8$ for points of interest near $x = -1$ like $x = -0.9997$. Such a task is not too terribly difficult, but it is tedious enough to be droll and time consuming (just imagine having to multiply -0.9997 by itself three times). The point of this lecture is to reduce this task to the rather simple task of evaluating $L(x) = 3x - 12$ at the points of interest.

Notice that $L(x)$ is a line. Specifically, $L(x)$ is the line tangent to $f(x)$ at $x = -1$ as shown in the graph below.



Notice, too, that $L(x)$ and $f(x)$ are nearly indistinguishable for *x*-values close to -1. Accordingly, if $a \approx -1$, then $L(a) \approx f(a)$.

The technique of using a tangent line to estimate values of a function is called linearization. In general, consider some function *f*. The line tangent to *f* at $x = a$ contains the point $(a, f(a))$ and has a slope of $m = f'(a)$. Substituting into the point-slope form of a linear equation, we obtain a function called the *linearization* of *f* at *a* as defined below.

$$y - y_1 = m(x - x_1)$$
$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$

The linearization of *f* at *a* is given by

$$L(x) = f(a) + f'(a)(x - a)$$

Differentials

To now, we have considered dy/dx, not as a ratio, but as notation for the derivative. Now, we will discuss linearization using what is called differential notation. In this discussion, both *dx* and *dy* are called *differentials*. The *dx* differential is an independent variable. The *dy* differential is defined in terms of *dx* as below.

The dy differential is defined in terms of dx by the equation.

$$dy = f'(a)\,dx.$$

The right triangle in Figure 1 illustrates the meaning of the differentials. The leg parallel to the x-axis represents dx. The leg parallel to the y-axis represents dy.

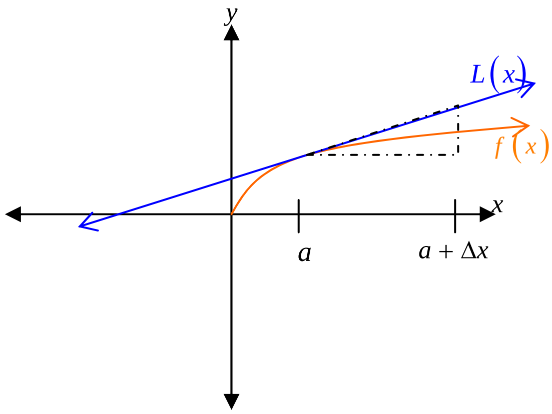


Figure 1

Notice $dx = \Delta x$, but $dy \neq \Delta y$. In other words, $dx = (a + \Delta x) - a$, but $dy \neq f(a + \Delta x) - f(a)$. Consequently, the differential dy is *not* a measure of the change in y that occurs in the function. Instead, dy is a measure of the change in y that occurs in the linearization of the function. That is, $dy = L(a + \Delta x) - L(a)$. Nevertheless, $dy \approx \Delta y$, and if placed in the position of knowing $f(a)$ and dy, we can estimate $f(a + dx)$ as $f(a) + dy$.

Consider the function $f(x) = \sqrt{x+5}$. Suppose we want to estimate $f(4.02)$. Note that we are interested in $x = 4.02$, so we let $a = 4$ and let $\Delta x = 0.02 = dx$. First, we calculate $f'(x)$ as below.

$$f'(x) = \frac{d}{dx}\left[f(x)\right]$$

$$f'(x) = \frac{d}{dx}\left(\sqrt{x+5}\right)$$

$$f'(x) = \frac{d}{dx}\left[(x+5)^{\frac{1}{2}}\right]$$

$$f'(x) = \frac{1}{2}(x+5)^{-\frac{1}{2}}\frac{d}{dx}(x+5)$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+5}}\cdot 1$$

$$f'(x) = \frac{1}{2\sqrt{x+5}}$$

Second, we calculate the differential *dy* using the equation $dy = f'(a)dx$ as below.

$$dy = \frac{1}{2\sqrt{(4)+5}} \cdot (0.02)$$

$$dy = \frac{1}{2\sqrt{9}} \cdot \frac{2}{100}$$

$$dy = \frac{1}{\cancel{2}(3)} \cdot \frac{\cancel{2}}{100}$$

$$dy = \frac{1}{300} = 0.00\overline{3}$$

Lastly, we can estimate $f(4.02)$ as $f(4) + 0.00\overline{3}$ to get $f(4) \approx 3.00\overline{3}$.

Lecture 13

Practice Problems

1st ed. problem set: Section 3.8 #1, #3, #13
2nd ed. problem set: Section 3.8 #1, #3, #17
3rd ed. problem set: Section 3.8 #5, #7, #25

Possible Exam Problems

#1 Given $f(x) = \sqrt[3]{x-3}$ estimate $f(11.078)$ using linearization. Choose a propitious *x*-value for $L(x)$.

Answer: Selecting $x = 11$, $L(x) = \frac{1}{12}x + \frac{13}{12}$.

Since $L(11.078) = 2.0065$, $f(11.078) \approx 2.0065$.

#2 Given $y = \cos(x)$, compute the differential dy in general terms.

Answer: $dy = -\sin(x)\,dx$.

Example Exercise

> Suppose $f(3) = 4$ and $f'(x) = \frac{x}{\sqrt{x^2+7}}$. Estimate the value of $f(3.01)$.

Find $L(x)$ using $a = 3$

$$L(x) = f(a) + f'(a)(x-a)$$
$$L(x) = 4 + \frac{3}{\sqrt{3^2+7}}(x-3)$$
$$L(x) \approx 4 + 0.75(x-3)$$
$$L(x) \approx 0.75x + 1.75$$

Evaluate $L(3.01)$.

$$L(x) \approx 0.75x + 1.75$$
$$L(3.01) \approx 0.75(3.01) + 1.75$$
$$L(3.01) \approx 4.0075$$

Assume the linearization function approximates $f(x)$.

$$f(3.01) \approx L(3.01)$$
$$f(3.01) \approx 4.0075$$

Application Exercise

In optics, *paraxial rays* are light rays that arrive at shallow angles relative to the optical axis. Suppose $y = \sin\theta$ calculates a light ray's path (where θ represents the rays angle of incident in relation to the optical axis). For any paraxial ray, $\theta \approx 0$. Let $\theta = 0$ and use the linearization for $y = \sin\theta$ to find a linearization function for approximating the path of paraxial rays.