

### Derivatives of Natural Logarithmic Functions

Somewhat surprisingly,  $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$  where  $x > 0$ . Employing the chain rule, we obtain the rule below.

*General Natural Logarithm Rule:* If  $f$  is differentiable at  $x$ , then

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

for all  $x$  where  $f(x) > 0$ .

For instance, let  $g(x) = \ln(x)$ ,  $f(x) = \sin x$ , and  $\beta(x) = g(f(x)) = \ln(\sin x)$ . Employing the chain rule,  $\beta' = g'(f(x)) \cdot f'(x)$ , so we have

$$\beta = \ln(\sin x)$$

$$\beta' = \frac{\frac{d}{dx}[\sin x]}{\sin x}$$

$$\beta' = \frac{\cos x}{\sin x}$$

$$\beta' = \cot x$$

### Logarithmic Differentiation

Calculating derivatives is often simplified using logarithms. Consider exponential functions of the form  $y = b^x$  where  $b$  is some positive number not equal to one. Finding  $y'$  is easier if we first take the natural log of both sides of the equation. In other words, we make the  $y$  and the  $b^x$  arguments of the natural log as below.

$$y = b^x$$

$$\ln(y) = \ln(b^x)$$

Recall that  $\log_b A^x = x \cdot \log_b A$ . Employing this property of logarithms, we can write:

$$\ln(y) = x \cdot \ln(b).$$

Differentiating implicitly with respect to  $x$ , we attain:

## Lecture 12

$$\ln(y) = x \cdot \ln(b)$$
$$\frac{1}{y} \frac{dy}{dx} = \ln b$$

Solving for  $dy/dx$  and substituting  $b^x$  for  $y$  we have the derivative.

$$\frac{dy}{dx} = y \cdot \ln b$$
$$\frac{dy}{dx} = b^x \cdot \ln b$$

Thus, we have a general rule for the derivative of exponential functions.

If  $y = b^x$  where  $b$  is a positive number not equal to one, then  $y' = b^x \cdot \ln b$ .

When we take logarithms before differentiating, the method is called *logarithmic differentiation*. In the example above, we used logarithmic differentiation to obtain our general rule for the derivative of exponential functions with base  $b$ , a positive number not equal to one. Suppose  $b = e$ , we obtain:  $y' = e^x \cdot \ln e$ , and since  $\ln e = 1$ , we have a rule for the derivative of  $y = e^x$ .

If  $y = e^x$ , then  $y' = e^x$ .

In the next example, we will use logarithmic differentiation to find the derivative of a scary-looking function. Consider the big, hairy polynomial below.

$$P(x) = (3x^2 + 5x + 1)^6 (7x^3 - x^2 + 2x + 1)^5$$

We could differentiate using the product rule and chain rule, but logarithmic differentiation is easier. First, we make both sides of the equation the argument of the natural log.

$$\ln[P(x)] = \ln\left[(3x^2 + 5x + 1)^6 (7x^3 - x^2 + 2x + 1)^5\right]$$

Second, we use properties of logs to write the right side as a sum of products as below.

$$\ln[P(x)] = 6\ln(3x^2 + 5x + 1) + 5\ln(7x^3 - x^2 + 2x + 1)$$

Third, we differentiate implicitly with respect to  $x$ .

Lecture 12

$$\frac{1}{P(x)} \frac{dy}{dx} = \frac{6(6x+5)}{3x^2+5x+1} + \frac{5(21x^2-2x+2)}{7x^3-x^2+2x+1}$$

$$\frac{dy}{dx} = P(x) \cdot \left[ \frac{6(6x+5)}{3x^2+5x+1} + \frac{5(21x^2-2x+2)}{7x^3-x^2+2x+1} \right]$$

Lastly, we substitute for  $y$  or, in this case, for  $P(x)$ .

$$\frac{dy}{dx} = (3x^2+5x+1)^6 (7x^3-x^2+2x+1)^5 \cdot \left[ \frac{6(6x+5)}{3x^2+5x+1} + \frac{5(21x^2-2x+2)}{7x^3-x^2+2x+1} \right]$$

In the last example, logarithmic differentiation saved us from having to use the product rule together with the chain rule. In the next example, logarithmic differentiation allows us to differentiate where our previous techniques do not apply. Consider  $y = (\sin x)^{x^2}$ . To differentiate, we make both sides the argument of the natural logarithm.

$$\ln(y) = \ln \left[ (\sin x)^{x^2} \right]$$

Using properties of logarithms, we write a pair of factors on the right side.

$$\ln(y) = x^2 \cdot \ln(\sin x)$$

Now, we differentiate implicitly with respect to  $x$ .

$$\frac{1}{y} \frac{dy}{dx} = x^2 \cdot \frac{d}{dx} [\ln(\sin x)] + \ln(\sin x) \cdot \frac{d}{dx} (x^2)$$

$$\frac{1}{y} \frac{dy}{dx} = x^2 \cdot \frac{1}{\sin x} \frac{d}{dx} (\sin x) + \ln(\sin x) \cdot 2x$$

$$\frac{dy}{dx} = y \cdot \left[ x^2 \cdot \frac{\cos x}{\sin x} + 2x \ln(\sin x) \right]$$

Substituting for  $y$ , we obtain  $dy/dx$ :

$$\frac{dy}{dx} = (\sin x)^{x^2} \cdot \left[ x^2 \cdot \frac{\cos x}{\sin x} + 2x \ln(\sin x) \right]$$

**Derivatives of Logarithmic Functions**

At the start of this lecture, we stated that  $\frac{d}{dx}[\ln x] = 1/x$ . What of  $\frac{d}{dx}[\log_b x]$ ? In other words, if  $y = \log_b x$ , then what is  $y'$ ? Well, if  $y = \log_b x$ , then we can use the definition of a logarithm to write:

$$b^y = x.$$

Differentiating implicitly, we can use the general rule for the derivative of exponential functions to write:

$$b^y \ln b \frac{dy}{dx} = 1.$$

Solving for  $dy/dx$  we have:

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}.$$

Recall that  $b^y = x$ , so we obtain:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

This gives us a general rule for the derivative of logarithmic functions.

If  $y = \log_b x$  where  $b$  is a positive number not equal to one and  $x > 0$ , then

$$y' = \frac{1}{x \ln b}.$$

### Practice Problems

- 1st ed. problem set: Section 3.7 #3–11 odd, #15–21 odd, #25–31 odd  
 2nd ed. problem set: Section 3.7 #3–13 odd, #17, #19, #27–33 odd  
 3rd ed. problem set: Section 3.7 #3–13 odd, #17, #19, #23, #27–35 odd

### Possible Exam Problems

#1 Given  $y = \ln(ax^n)$ , find  $y'$ .

$$\text{Answer: } y' = \frac{nax^{n-1}}{ax^n} = \frac{n}{x}$$

#2 Given that  $\ln|x| = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$ , find  $\frac{d}{dx}[\ln|x|]$ .

$$\text{Answer: } \frac{d}{dx}[\ln|x|] = \frac{1}{x} \text{ where } x \neq 0$$

#3 Given  $y = 2^x$ , find  $y'$ .

$$\text{Answer: } y' = 2^x \cdot (\ln 2)$$

#4 Given  $y = (17x^2 + 1)^4 (x^3 - 13x + 5)^4$ , find  $y'$ .

$$\text{Answer: } y' = 4(17x^2 + 1)^4 (x^3 - 13x + 5)^4 \left[ \frac{34x}{17x^2 + 1} + \frac{3x^2 - 13}{x^3 - 13x + 5} \right]$$

#5 Given  $f(x) = \log_2 x$ , find  $f'$ .

$$\text{Answer: } y' = \frac{1}{x \ln 2}$$

**Example Exercise 1**

Suppose  $y = 5^x$ . Use logarithmic differentiation to find  $y'$ .

Take the natural log of both sides.

$$y = 5^x$$
$$\ln(y) = \ln(5^x)$$

Apply the property of logs that states  $\log_b(a^x) = x \log_b(a)$ .

$$\ln(y) = \ln(5^x)$$
$$\ln(y) = x \ln(5)$$

Differentiate implicitly. Note that  $\ln(5)$  is a constant, being a specific number.

$$\ln(y) = x \ln(5)$$
$$\frac{1}{y} y' = \ln(5)$$
$$y' = y \cdot \ln(5)$$

Recall  $y = 5^x$  and substitute.

$$y' = y \cdot \ln(5)$$
$$y' = 5^x \cdot \ln(5)$$

**Example Exercise 2**

Suppose  $y = (3x^2 + 4)^5 (2x^4 - 6)^4$ . Find  $y'$ .

Take the natural log of both sides.

$$y = (3x^2 + 4)^5 (2x^4 - 6)^4$$

$$\ln(y) = \ln\left[(3x^2 + 4)^5 (2x^4 - 6)^4\right]$$

Use properties of logs to expand the right side of the equation.

$$\ln(y) = \ln\left[(3x^2 + 4)^5 (2x^4 - 6)^4\right]$$

$$\ln(y) = \ln\left[(3x^2 + 4)^5\right] + \ln\left[(2x^4 - 6)^4\right]$$

$$\ln(y) = 5 \ln\left[(3x^2 + 4)\right] + 4 \ln\left[(2x^4 - 6)\right]$$

Differentiate implicitly.

$$\ln(y) = 5 \ln\left[(3x^2 + 4)\right] + 4 \ln\left[(2x^4 - 6)\right]$$

$$\frac{1}{y} y' = 5 \frac{6x}{3x^2 + 4} + 4 \frac{8x}{2x^4 - 6}$$

$$y' = y \left[ \frac{30x}{3x^2 + 4} + \frac{32x}{2x^4 - 6} \right]$$

Recall  $y = (3x^2 + 4)^5 (2x^4 - 6)^4$  and substitute then simplify.

$$y' = y \left[ \frac{30x}{3x^2 + 4} + \frac{32x}{2x^4 - 6} \right]$$

$$y' = (3x^2 + 4)^5 (2x^4 - 6)^4 \left[ \frac{30x}{3x^2 + 4} + \frac{32x}{2x^4 - 6} \right]$$

$$y' = 2x(3x^2 + 4)^5 (2x^4 - 6)^4 \left[ \frac{15}{3x^2 + 4} + \frac{16}{2x^4 - 6} \right]$$

**Example Exercise 3**

Suppose  $y = (\sin x)^{x^2}$ . Find  $\frac{dy}{dx}$ .

Take the natural log of both sides.

$$y = (\sin x)^{x^2}$$

$$\ln(y) = \ln\left[(\sin x)^{x^2}\right]$$

Apply the property of logs that states  $\log_b(a^x) = x \log_b(a)$ .

$$\ln(y) = \ln\left[(\sin x)^{x^2}\right]$$

$$\ln(y) = x^2 \ln(\sin x)$$

Differentiate implicitly.

$$\ln(y) = x^2 \ln(\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = x^2 \frac{d}{dx}[\ln(\sin x)] + \ln(\sin x) \frac{d}{dx}[x^2]$$

$$\frac{1}{y} \frac{dy}{dx} = x^2 \frac{\cos x}{\sin x} + 2x \ln(\sin x)$$

Solve for  $\frac{dy}{dx}$  then substitute for  $y$  using the original function.

$$\frac{1}{y} \frac{dy}{dx} = x^2 \frac{\cos x}{\sin x} + 2x \ln(\sin x)$$

$$\frac{dy}{dx} = y \left[ x^2 \frac{\cos x}{\sin x} + 2x \ln(\sin x) \right]$$

$$\frac{dy}{dx} = (\sin x)^{x^2} \left[ x^2 \frac{\cos x}{\sin x} + 2x \ln(\sin x) \right]$$

### Application Exercise

In optics, the transparency  $T$  of a substance equals the ratio of the intensity of light transmitted through the substance to the intensity of incident light where incident light is the light falling on the surface of a substance. The opacity of a substance equals the reciprocal of transparency. The density  $D$  of a substance equals the common logarithm of the opacity.

Find density in terms of transparency then find  $dD/dT$  .