

Implicit Differentiation

The functions differentiated in the previous lectures have been nice, neat functions, the kind we do not mind introducing to dear old mum. We might call them explicit functions (although nobody does) because the rule that assigns a y -value to any given x -value is explicit—that is, straightforward and obvious. Consider $y = 1/x$, which makes y 's dependency on x obvious: for any x -value, the corresponding y -value is the quotient of 1 and the particular x -value. The same function can be defined less explicitly (or implicitly) as $xy = 1$. Here the x and y become all tangled up by the multiplication operation. Of course, with $xy = 1$ a simple algebraic transformation of the equation yields $y = x^{-1}$, which is differentiable with the power rule. Sometimes, however, functions are stated in such a tangled mess that it becomes difficult to untangle by isolating y on one side of the equation. Fortunately, the chain rule gives us a procedure that enables us to find the derivative using the tangled mess. This procedure is called implicit differentiation.

Consider the function implicit in the equation $y^7 + y + x^5 + 13x^3 = \sin(y)$. To differentiate implicitly, we differentiate the entire equation with respect to x . Accordingly, we treat x as a variable as usual, but we will think of y as a function of x such that its derivative will be dy/dx . Differentiating each term with respect to x , we obtain:

$$7y^6 \frac{dy}{dx} + \frac{dy}{dx} + 5x^4 + 39x^2 = \cos(y) \frac{dy}{dx}.$$

Notice that we employ the chain rule and obtain $7y^6 dy/dx$ when we differentiate y^7 . Similarly, when we differentiate $\sin(y)$, we get $\cos(y)dy/dx$. Now, we finish by solving the equation for dy/dx as below.

$$\begin{aligned} 7y^6 \frac{dy}{dx} + \frac{dy}{dx} - \cos(y) \frac{dy}{dx} &= -5x^4 - 39x^2 \\ \frac{dy}{dx} [7y^6 + 1 - \cos(y)] &= -5x^4 - 39x^2 \\ \frac{dy}{dx} &= \frac{-5x^4 - 39x^2}{7y^6 + 1 - \cos(y)} \end{aligned}$$

Implicit differentiation can be employed to differentiate inverse trigonometric functions. Recall that if $y = \sin^{-1}(x)$, then $\sin(y) = x$ where $-\pi/2 \leq y \leq \pi/2$, and consider the differentiation of $y = \sin^{-1}(x)$ below.

$$\begin{aligned} \sin(y) &= x \\ \cos(y) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \end{aligned}$$

Lecture 11

From the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$, we know $\cos(y) = \sqrt{1 - \sin^2(y)}$ for non-negative values of $\cos(y)$, which is the case for the interval $-\pi/2 \leq y \leq \pi/2$. Recall that we started with $\sin(y) = x$. Substituting x for $\sin(y)$, we obtain $\cos(y) = \sqrt{1 - x^2}$, which can, in turn, be substituted into our expression for dy/dx giving us a rule for the derivative of the inverse sine function.

$$\frac{d}{dx} [\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

We can find the derivative of the arctangent function in a similar manner.

$$y = \tan^{-1} x$$

$$\tan(y) = x$$

$$\sec^2(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{1+x^2}$$

Practice Problems

- 1st ed. problem set: Section 3.6 #3–9 odd, #11, #15, #25–29 odd
2nd ed. problem set: Section 3.6 #3–11 odd, #13, #17, #27–31 odd
3rd ed. problem set: Section 3.6 #3–11 odd, #15–19 odd, #29–33 odd

Possible Exam Problems

#1 Given $f(x) = \sin^{-1}(x)$, find the line tangent to f at $x = 1/2$.

$$\text{Answer: } y = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}} + \frac{\pi}{6}$$

#2 Differentiate $x^2y - x^3 = e^y$ implicitly with respect to x .

$$\begin{aligned} \text{Answer: } x^2 \cdot \frac{dy}{dx} + y \cdot 2x - 3x^2 &= e^y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{3x^2 - 2xy}{x^2 - e^y} \end{aligned}$$

Example Exercise 1

Suppose $y = \arccos x$. Find $\frac{dy}{dx}$.

Recall that if $y = \cos^{-1}(x)$, then $\cos(y) = x$ where $0 \leq y \leq \pi$, and consider the differentiation of $y = \cos^{-1}(x)$ below.

$$y = \cos^{-1} x$$

$$\cos(y) = x$$

$$-\sin(y) \frac{dy}{dx} = 1$$

$$\sin(y) \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{\sin(y)}$$

Recall the Pythagorean Identity: $\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \sqrt{1 - \cos^2 y}$, and substitute for $\sin y$ above.

$$\frac{dy}{dx} = \frac{-1}{\sin(y)}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \cos^2 y}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

Example Exercise 2

Suppose x and y are related by the equation $y \tan(x + y) = 4$. Find $\frac{dy}{dx}$.

Differentiate implicitly.

$$y \tan(x + y) = 4$$

$$y \cdot \frac{d}{dx}(\tan(x + y)) + \tan(x + y) \cdot \frac{dy}{dx} = 0$$

$$y \cdot \sec^2(x + y) \cdot \frac{d}{dx}(x + y) + \tan(x + y) \cdot \frac{dy}{dx} = 0$$

$$y \cdot \sec^2(x + y) \cdot \left(1 + \frac{dy}{dx}\right) + \tan(x + y) \cdot \frac{dy}{dx} = 0$$

Solve for dy/dx .

$$y \cdot \sec^2(x + y) \cdot \left(1 + \frac{dy}{dx}\right) + \tan(x + y) \cdot \frac{dy}{dx} = 0$$

$$y \cdot \sec^2(x + y) + y \cdot \sec^2(x + y) \cdot \frac{dy}{dx} + \tan(x + y) \cdot \frac{dy}{dx} = 0$$

$$y \cdot \sec^2(x + y) \cdot \frac{dy}{dx} + \tan(x + y) \cdot \frac{dy}{dx} = -y \cdot \sec^2(x + y)$$

$$\frac{dy}{dx} [y \cdot \sec^2(x + y) + \tan(x + y)] = -y \cdot \sec^2(x + y)$$

$$\frac{dy}{dx} = \frac{-y \cdot \sec^2(x + y)}{y \cdot \sec^2(x + y) + \tan(x + y)}$$

Note that the original equation implies $y = 4/\tan(x + y)$. Use this fact along with trigonometric identities to simplify the answer.

$$\frac{dy}{dx} = -\frac{4 + 4 \tan^2(x + y)}{4 + 5 \tan^2(x + y)}$$

Example Exercise 3

Consider the curve described by the conditions on x and y below.

$$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{2^2}, \quad -2 \leq x \leq 2, \quad -2 \leq y \leq 2$$

Determine the points on the curve for which there is no tangent line with a defined slope.

Rewrite the equation using fractional exponents.

$$\begin{aligned} \sqrt[3]{x^2} + \sqrt[3]{y^2} &= \sqrt[3]{2^2} \\ x^{\frac{2}{3}} + y^{\frac{2}{3}} &= 2^{\frac{2}{3}} \end{aligned}$$

Differentiate implicitly.

$$\begin{aligned} x^{\frac{2}{3}} + y^{\frac{2}{3}} &= 2^{\frac{2}{3}} \\ \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \frac{dy}{dx} &= 0 \\ \frac{2}{3x^{\frac{1}{3}}} + \frac{2}{3y^{\frac{1}{3}}} \frac{dy}{dx} &= 0 \\ \frac{2}{3\sqrt[3]{x}} + \frac{2}{3\sqrt[3]{y}} \frac{dy}{dx} &= 0 \end{aligned}$$

Note that the equation is undefined if $x = 0$ or $y = 0$. Substitute these values into the original equation to find the corresponding points where the curve is non-differentiable.

$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{2^2}$	$\sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{2^2}$
$\sqrt[3]{0^2} + \sqrt[3]{y^2} = \sqrt[3]{2^2}$	$\sqrt[3]{x^2} + \sqrt[3]{0^2} = \sqrt[3]{2^2}$
$\sqrt[3]{y^2} = \sqrt[3]{2^2}$	$\sqrt[3]{x^2} = \sqrt[3]{2^2}$
$y^2 = 2^2$	$x^2 = 2^2$
$y^2 = 4$	$x^2 = 4$
$y = \pm 2$	$x = \pm 2$
$(0, \pm 2)$	$(\pm 2, 0)$

Application Exercise

Suppose that the equation $y^2 + 9 = z^2$ (where y represents the altitude of a rocket and z represents the distance between rocket and a tracking radar station) implicitly relates the position of a rocket to the position of a radar station at any time t since launch. Implicitly differentiate the equation with respect to time then calculate the velocity of the rocket given the radar readouts below.

<u>Altitude of rocket:</u>	<u>4 miles</u>
<u>Distance between rocket and radar:</u>	<u>5 miles</u>
<u>Rate at which distance between rocket and radar is changing over time:</u>	<u>5000 mph</u>