

Differentiation with Trigonometric & e^x Functions

Previous lectures have dealt rather specifically with common algebraic functions. This lecture will provide differentiation rules for trigonometric, exponential, and logarithmic functions.

Remarkably, the derivative of the sine-wave function is the cosine-wave function, and the derivative of the cosine-wave function is the sine-wave function reflected over the x -axis. Using these two facts we can find rules for the remaining trigonometric functions. For instance, consider $y = \tan(x)$. We can find dy/dx using the quotient rule.

$$\begin{aligned}
 y &= \tan(x) \\
 y' &= \frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\
 y' &= \frac{\cos(x) \cdot \frac{d}{dx}[\sin(x)] - \sin(x) \cdot \frac{d}{dx}[\cos(x)]}{[\cos(x)]^2} \\
 y' &= \frac{\cos(x) \cdot \cos(x) + \sin(x) \cdot \sin(x)}{\cos^2(x)} \\
 y' &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
 y' &= \frac{1}{\cos^2(x)} \\
 y' &= \sec^2(x)
 \end{aligned}$$

Using similar techniques, we arrive at the derivatives for the trigonometric functions given below.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

Using the chain rule with $\frac{d}{dx}[\sin(x)] = \cos(x)$, we attain a general rule for sine functions.

General Sine Function Rule: If f is differentiable at x , then

$$\frac{d}{dx}[\sin(f(x))] = \cos(f(x)) \cdot f'(x).$$

Using the chain rule with $\frac{d}{dx}[\cos(x)] = -\sin(x)$, we attain the general rule for cosine functions.

General Cosine Function Rule: If f is differentiable at x , then

$$\frac{d}{dx}[\cos(f(x))] = -\sin(f(x)) \cdot f'(x).$$

For example, if $y = \cos(14x^2)$, then $y' = -28x \sin(14x^2)$.

Turning our attention now to the natural exponential function, $y = e^x$, we have another remarkable fact, namely, the derivative of e^x is e^x . Yes, that's correct; e^x is its own derivative. Coupled with the chain rule, we have the general rule below.

General $e^{f(x)}$ Rule: If f is differentiable at x , then

$$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} \cdot f'(x).$$

Practice Problems in *Calculus: Concepts and Contexts* by James Stewart

- 1st ed. problem set: Section 3.2 #3, #5, #13, #21a
 Section 3.4 #1–17 odd, #19a, #25, #29a
 Section 3.5 #3, #5, #9, #13, #17, #23, #25, #29, #33a
- 2nd ed. problem set: Section 3.2 #3, #5, #13, #19, #25a
 Section 3.4 #1–17 odd, #19a, #21a, #23, #29a
 Section 3.5 #1, #5, #11, #13, #15, #19, #25, #29, #31, #33a
- 3rd ed. problem set: Section 3.2 #3, #5, #13, #21, #25a, #27a
 Section 3.4 #1–17 odd, #19a, #21a, #23, #31a
 Section 3.5 #1, #5, #11, #15, #19, #23, #29, #31, #33, #55, #57

Practice Problems in *Calculus: Early Transcendentals* by Briggs and Cochran

- 1st ed. problem set: 3.2 #23, #37, #44
 3.3 #37-41 odd, #55
 3.4 #15-29 odd
 3.6 #11-15 odd, #21-29 odd

Possible Exam Problems

#1 Given $f(x) = e^{\cot(5x)}$, find $f'(x)$.

Answer: $f'(x) = -5e^{\cot(5x)} \csc^2(5x)$

#2 Given $g(x) = \sin(5x)$, find $g^{11}(x)$.

Answer: $g^{11}(x) = -5^{11} \cos(5x)$

#3 Find a non-trivial solution for the differential equation $y' - y = 0$.

Answer: $y = e^x$

#4 Find a non-trivial solution for the differential equation $y'' + y = 0$.

Answer: $y = \sin x$ or $y = \cos x$

#5 Show that $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ using quotient rule and the fact that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Answer:
$$\begin{aligned} \frac{d}{dx}[\sec(x)] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{\cos x \cdot \frac{d}{dx}[1] - \left[1 \cdot \frac{d}{dx}\cos(x)\right]}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \\ &= \tan(x) \cdot \sec(x) \end{aligned}$$

Example Exercise 1

Suppose $f(x) = \frac{\cos x}{1 + \sin x}$. Find $f'(x)$.

Apply the quotient rule.

$$f(x) = \frac{\cos x}{1 + \sin x}$$

$$f'(x) = \frac{(1 + \sin x) \cdot \frac{d}{dx}(\cos x) - \left[\cos x \cdot \frac{d}{dx}(1 + \sin x) \right]}{(1 + \sin x)^2}$$

$$f'(x) = \frac{(1 + \sin x) \cdot -\sin x - [\cos x \cdot (0 + \cos x)]}{(1 + \sin x)^2}$$

$$f'(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

Recall the Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$.

$$f'(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$f'(x) = \frac{-\sin x - 1(\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}$$

$$f'(x) = \frac{-\sin x - 1}{(1 + \sin x)^2}$$

$$f'(x) = \frac{-1(\sin x + 1)}{(1 + \sin x)^2}$$

$$f'(x) = \frac{-1(\cancel{\sin x + 1})}{(1 + \sin x)(\cancel{1 + \sin x})}$$

$$f'(x) = -\frac{1}{(1 + \sin x)}$$

Example Exercise 2

Suppose $y = \sin(2x) + \cos^2 x$. Find dy/dx .

Apply the General Sine Function Rule and the Chain Rule and simplify.

$$\begin{aligned}y &= \sin(2x) + (\cos x)^2 \\y' &= \cos(2x) \cdot \frac{d}{dx}(2x) + 2(\cos x) \cdot \frac{d}{dx}(\cos x) \\y' &= \cos(2x) \cdot 2 + 2(\cos x) \cdot -\sin x \\y' &= 2\cos(2x) - 2\sin x \cdot \cos x\end{aligned}$$

Apply the Double-Angle Identity.

$$\begin{aligned}y' &= 2\cos(2x) - 2\sin x \cdot \cos x \\y' &= 2\cos(2x) - \sin(2x)\end{aligned}$$

Example Exercise 3

Suppose $y = \tan(\sin x)$. Find dy/dx .

Apply chain rule and simplify.

$$\begin{aligned}y &= \tan(\sin x) \\y' &= \sec^2(\sin x) \cdot \frac{d}{dx}(\sin x) \\y' &= \sec^2(\sin x) \cdot \cos x\end{aligned}$$

Example Exercise 4

Suppose $y = 5x^2 e^{x^2}$. Find dy/dx .

Apply product rule.

$$y = 5x^2 e^{x^2}$$

$$y' = 5x^2 \cdot \frac{d}{dx}(e^{x^2}) + e^{x^2} \cdot \frac{d}{dx}(5x^2)$$

Apply the General $e^{f(x)}$ Rule along with the Power Rule.

$$y' = 5x^2 \cdot \frac{d}{dx}(e^{x^2}) + e^{x^2} \cdot \frac{d}{dx}(5x^2)$$

$$y' = 5x^2 \cdot e^{x^2} \cdot \frac{d}{dx}(x^2) + e^{x^2} \cdot 10x$$

$$y' = 5x^2 \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot 10x$$

Simplify and factor the greatest common factor.

$$y' = 5x^2 \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot 10x$$

$$y' = 10x^3 \cdot e^{x^2} + 10xe^{x^2}$$

$$y' = 10xe^{x^2} (x^2 + 1)$$

Example Exercise 5

Suppose $y = \sec x$. Determine when the curve has a horizontal tangent.

Find y' .

$$y = \sec x$$

$$y' = \sec x \tan x$$

Set y' equal to zero and solve.

$$\sec x \tan x = 0$$

$$\sec x \neq 0 \quad \tan x = 0$$

$$x = 0 + k\pi$$

Example Exercise 6

Suppose $f(x) = e^{\sin x}$. Find the line tangent to $f(x)$ when $x = \pi/6$.

Find the point of tangency by evaluating $f(\pi/6)$.

$$f(\pi/6) = e^{\sin(\pi/6)}$$

$$f(\pi/6) = e^{\frac{1}{2}}$$

$$f(\pi/6) = \sqrt{e}$$

Find $f'(x)$ using the General $e^{f(x)}$ Rule.

$$f(x) = e^{\sin(x)}$$

$$f'(x) = e^{\sin(x)} \cdot \frac{d}{dx}(\sin x)$$

$$f'(x) = e^{\sin(x)} \cdot \cos x$$

Find the slope of the line tangent to $f(x)$ when $x = \pi/6$.

$$f'(x) = e^{\sin(x)} \cdot \cos x$$

$$f'(\pi/6) = e^{\sin(\pi/6)} \cdot \cos(\pi/6)$$

$$f'(\pi/6) = e^{\frac{1}{2}} \cdot \frac{\sqrt{3}}{2} = \sqrt{e} \cdot \frac{\sqrt{3}}{2}$$

$$f'(\pi/6) = \frac{\sqrt{3e}}{2}$$

Substitute into the point-slope formula for linear equations.

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{e} = \frac{\sqrt{3e}}{2} \left(x - \frac{\pi}{6} \right)$$

$$y - \sqrt{e} = \frac{\sqrt{3e}}{2} x - \frac{\pi\sqrt{3e}}{12}$$

$$y = \frac{\sqrt{3e}}{2} x - \frac{\pi\sqrt{3e}}{12} + \sqrt{e}$$

Example Exercise 7

Suppose $y = e^{x^2+3x}$. Determine when the curve has a horizontal tangent.

Find y' using the General $e^{f(x)}$ Rule.

$$y = e^{x^2+3x}$$

$$y' = e^{x^2} \cdot \frac{d}{dx}(x^2 + 3x)$$

$$y' = e^{x^2} \cdot (2x + 3)$$

Set y' equal to zero and solve.

$$(2x+3)e^{x^2} = 0$$

$$2x+3=0 \quad e^{x^2} \neq 0$$

$$2x = -3$$

$$x = -\frac{3}{2}$$

Application Exercise

Oscillation is motion that repeats after a *period* of elapsed time. Simple harmonic motion is a special type of oscillation. The function $x(t) = A \sin(\omega t + \delta)$ models simple harmonic motion over time t where ω represents a constant angular speed of the particle in motion and δ represents some initial angular position of the particle at time $t = 0$. Let $x(t)$ represent the displacement of a particle obeying simple harmonic motion. Find the velocity of the particle at time t .