Lecture 1

Calculus

What is calculus? Calculus is a branch of mathematics derived from two problems: finding instantaneous rate of change and the area between a curve and the *x*-axis. It turns out, these two problems are related. We call the solution to one the derivative and the solution to the other the anti-derivative or integral. They are tied together—believe it or not—by the *Fundamental Theorem of Calculus*. Both problems, the derivative and integral, are founded on the idea of something called a limit, and, as a consequence, our discussion starts with limits.

Limits

What is a limit? In common usage, a *limit* is something that is frequently not reached, though it might be with mundane or dire consequences. Think of our personal drink limit at a frat party or rave. In this case, it is best if we don't reach our limit, for, if we do, we might awaken to find ourselves in what could be a surreal "What Happens in Vegas, Stays in Vegas" commercial.

Similarly, in mathematics, a limit is a number that is sometimes not reached, though it might be. In this case, however, the consequences are never dire. Of course, an astute reader has noticed the passive voice and is wondering, "What's doing the 'reaching'?" Some particular function is "reaching." Consider the notation below.

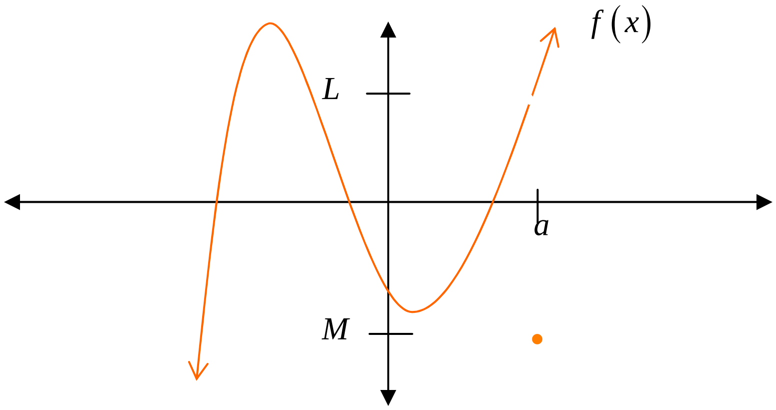
$$\lim_{x \to a} f(x)$$

This notation reads "the limit of function *f* as *x* approaches *a*." So, the limit is associated with two things: some function *f* and some number *a*. We must stress, however, that the limit is *not* interested in $f(a)$. Rather the limit is a number that *f* approaches as *x*-values approach *a*. Formally, we define the limit as below.

> The *limit* of $f(x)$ as *x* approaches *a*, denoted $\lim_{x \to a} f(x)$, equals a number *L* if the values of *f* become arbitrarily close to *L* as *x* approaches sufficiently close to *a* from both sides on the number line.

The phrase "arbitrarily close to *L*" implies that anyone—including capricious pirate captains and whimsical princesses—can pick some arbitrary degree of closeness and eventually be satisfied by evaluating the function at some value sufficiently close to *a*. Imagine that our capricious pirate captain evaluates *f* at $a + 0.000000001$ and at $a - 0.0003$ and finds that *f* in both cases comes within one-billionth of *L*. Our pirate captain may say, "I am satisfied that *L* is the limit of *f* as *x* approaches *a*." Nevertheless, *L* is not the limit unless every other arbitrary test is also satisfied. Our whimsical princess has the right to interject, "Whoa! Just one minute! One-billionth is no satisfactory measure of proximity. I will not be satisfied unless I can get within one-godzillionth." And if it cannot be shown that *f* comes within one-godzillionth for some *x*-value very close to *a* (from both sides) then *L* is not the limit.

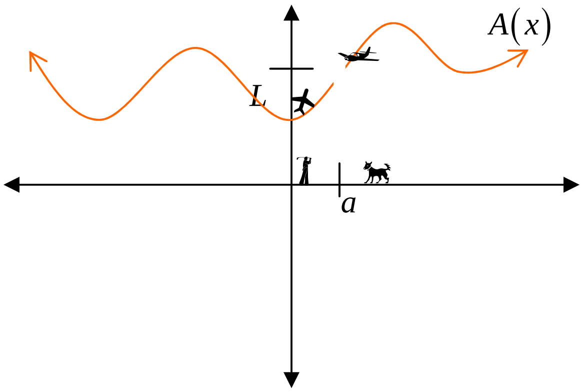
Notice all the discussion centered around getting "very close to *a*." It bears repeating, that the value of the function actually at *a*, i.e., $f(a)$, is irrelevant. Consider the function below.



For $f(x)$ above, $f(a)=M$, but the limit as *x*-approaches *a* is the number *L*, not the number *M*.

Besides the emphasis on "arbitrary" and "sufficiently close to *a*," the definition also stresses the idea of approaching *a* from both sides of the number line. If we were to worry only about one side, we would be discussing an entirely different creature than the limit, a creature called a one-sided limit. (Hopefully, the reader can guess what kind of creature it is. That's right. One-sided limits are also numbers.)

Before we define one-sided limits, let's illustrate what we mean by approaching *a* from both sides. Consider the graph of *A*(*x*) below. Imagine the function as a piece-wise function. One part of the function determines the flight path (or altitude) of some aircraft that must hover at varying altitudes above a man walking left to right along the *x*-axis. The other part of the function determines the flight path (or altitude) of another aircraft that must hover at varying altitudes above a dog walking right to left along the *x*-axis. If both aircraft are headed toward *L* as they approach *a*, then the limit of *f* at *a* is *L*.

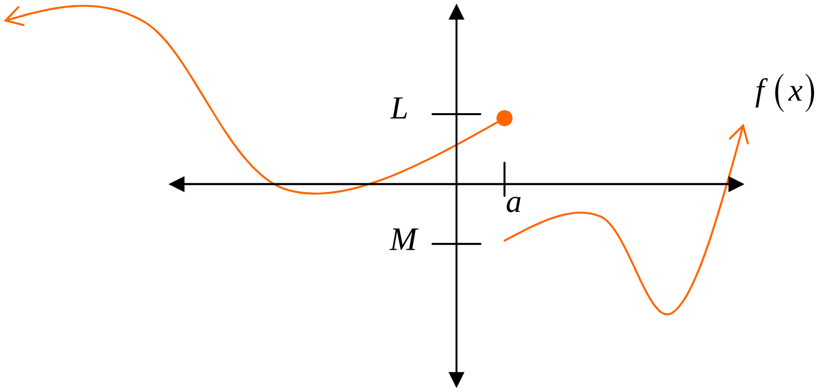


If the man's aircraft heads toward *L* and the dog's aircraft heads toward some number *M*, not equal to *L*, then the limit of *f* at *a* "does not exist," denoted DNE. To state this formally, we need to define *one-sided limits*. One-sided limits are *left-hand* and *right-hand limits* as defined below.

The *left-hand limit* of f as x approaches a, denoted $\lim_{x\to a^-} f(x)$, equals a number L if the values of f become arbitrarily close to L as x approaches sufficiently close to a from the left side of the number line.

The *right-hand limit* of f as x approaches a, denoted $\lim_{x\to a^+} f(x)$, equals a number L if the values of f become arbitrarily close to L as x approaches sufficiently close to a from the right side of the number line.

Consider the graph of $f(x)$ below. According to the definitions above, the left-hand limit of f as x approaches a is L. The right-hand limit of f as x approaches a is M.



Comparing the definition of a limit to the definition of one-sided limits, yields the following theorem.

$$\boxed{\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.}$$

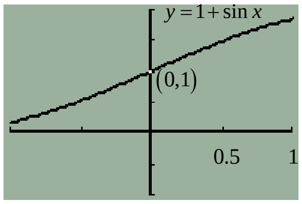
In other words, the limit of $f(x)$ as x approaches a exists and equals L only if each one-sided limit of $f(x)$ as x approaches a equals L.

Strict Limits

In the remainder of the lecture, we redefine limits more rigorously. Before redefining limits, let us consider $y(x) = \sin(x) + 1$ and the limit of this function as x approaches zero. In the process of considering $\lim_{x\to 0}(1+\sin x)$, we will attempt to find a positive real number δ such that whenever $|x-0| < \delta$ then $|y(x)-1| < \varepsilon$ for some given positive real number ε.

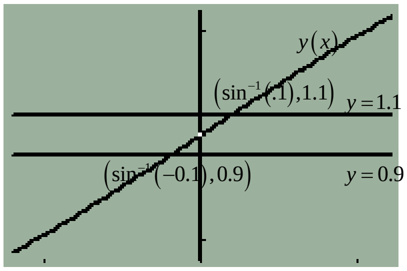
The expression, $|y(x)-1| < \varepsilon$, indicates the distance between $y(x)$ and one is smaller than ε. The expression, “whenever $|x-0| < \delta$,” indicates that $|y(x)-1| < \varepsilon$ is true for all values of x between zero and δ.

The two expressions, $|x-0|<\delta$ and $|y(x)-1|<\varepsilon$, place an emphasis on the point (0,1) because they indicate distances from zero on the *x*-axis and distances from one on the *y*-axis.



To find δ, we note that we can rewrite $|y(x)-1|<\varepsilon$ as $-\varepsilon<y(x)-1<\varepsilon$, which gives us $-\varepsilon+1<y(x)<\varepsilon+1$.

Assigning $\varepsilon=0.1$, we have $0.9<y(x)<1.1$. Hence, we draw horizontal lines $y=0.9$ and $y=1.1$, and note the intersection of $y(x)$ with these two horizontal lines on the graph below.



Using the graph, we see that any number along the interval $\sin^{-1}(-0.1)<x<\sin^{-1}(0.1)$ satisfies the conditions for δ, namely that $|y(x)-1|<0.1$ whenever $|x-0|<\delta$. Since sine is an odd function, we can replace $\sin^{-1}(-0.1)$ with $-\sin^{-1}(0.1)$ to obtain, $-\sin^{-1}(0.1)<x<\sin^{-1}(0.1)$. Hence, we know $|y(x)-1|<0.1$ whenever $|x-0|<\sin^{-1}(0.1)$, and δ equals any *x*-value that satisfies the inequality $|x|<\sin^{-1}(0.1)$.

Repeating this procedure requiring ever smaller values for ε, we find ever smaller values for δ, which indicates that $\lim_{x\to 0} y(x)=1$ and illustrates the following definition for the limit of a function.

Let f be a function defined on some interval *I* that contains the number *a* except possibly at *a* itself.

The **limit of** $f(x)$ **as** ***x*** **approaches** ***a***, denoted $\lim_{x\to a} f(x)$, is defined as the number *L* if for every positive real number ε there is a corresponding positive number δ such that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$.

The reader should note that the δ value might be smaller or larger than the given ε value, but that as successive smaller values for ε are selected the interval of allowed values for δ will shorten.

Limits at Infinity

Limits at infinity possess a strict definition analogous to the definition on the preceding page.

Let f be a function defined on some interval (a,∞). The **limit of $f(x)$ as x approaches infinity**, denoted $\lim\limits_{x\to\infty} f(x)$, is defined as the number L if for every positive real number ε there is a corresponding positive number N such that $\left|f(x)-L\right|<\varepsilon$ whenever $x>N$.

Consider the function $f(x)=1/x$ along the interval of positive x-values. To show that $\lim\limits_{x\to\infty} f(x)=0$, we need to show the existence of a positive real number N for any given positive number ε such that $\left|f(x)-0\right|<\varepsilon$ whenever $x>N$. Accordingly, we have the following.

$$\left|\frac{1}{x}-0\right|<\varepsilon$$

$$\left|\frac{1}{x}\right|<\varepsilon$$

Since $x > 0$ and $\varepsilon>0$, we obtain the following.

$$\frac{1}{x}<\varepsilon$$

$$1<\varepsilon x$$

$$\frac{1}{\varepsilon}<x$$

$$x>\frac{1}{\varepsilon}$$

Hence, for any positive number ε there is a corresponding number $N=1/\varepsilon$ such that $\left|f(x)-L\right|<\varepsilon$ whenever $x>N$. Therefore, $\lim\limits_{x\to\infty} f(x)=0$, and reversing the steps completes the proof.

Lecture 1

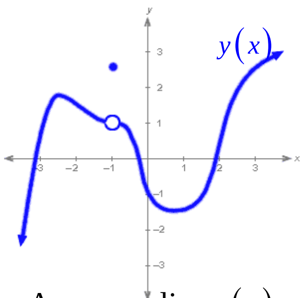
Practice Problems

1st ed. problem set:	2.2 #1–4 all & #7 also Appendix D #3, #5, #9, #13
2nd ed. problem set:	2.2 #1–7 all & #9 also Appendix D #3, #5, #9, #13
3rd ed. problem set:	2.2 #1–7 all plus #9 & #11 also Appendix D #3, #5, #9, #13

Possible Exam Problems

#1 Does the notation $\lim_{x\to a} f(x)$ refer to a number or a function? Answer: Number.

#2 Given the graph of $y(x)$, evaluate $\lim_{x\to -1} y(x)$.



Answer: $\lim_{x\to -1} y(x) = 1$.

#3 Find the values of δ such that $\left|\frac{1}{x} - 0.5\right| < 0.2$ whenever $|x-2| < \delta$. Answer: $\delta \le 4/7$.

#4 Show that $\lim_{x\to 2}(3x-1) = 5$.

Answer: We need to show that there exists δ corresponding to any given ε such that whenever $|x-2| < \delta$ then $|(3x-1)-5| < \varepsilon$:

$$
\begin{aligned}
&|(3x-1)-5| < \varepsilon \\
&|3x-6| < \varepsilon \\
&|3(x-2)| < \varepsilon \\
&|3||x-2| < \varepsilon \\
&3|x-2| < \varepsilon \\
&|x-2| < \varepsilon/3
\end{aligned}
$$

Since $|x-2| < \delta$, by substitution we see $|x-2| < \varepsilon/3 \Rightarrow \delta < \varepsilon/3$. Reversing the steps completes the proof.

#5 Show that $\lim\limits_{x\to\infty}\left(\dfrac{2x^3+1}{1-x^3}\right)=-2$.

Answer: We need to show $\exists$ some positive number N corresponding any given positive number ε such that $\left|\left(\dfrac{2x^3+1}{1-x^3}\right)-(-2)\right|<\varepsilon$ whenever $x>N$.

$$\left|\left(\frac{2x^3+1}{1-x^3}\right)-(-2)\right|<\varepsilon$$

$$\left|\left(\frac{2x^3+1}{1-x^3}\right)+\frac{2(1-x^3)}{1-x^3}\right|<\varepsilon$$

$$\left|\frac{2x^3+1+2-2x^3}{1-x^3}\right|<\varepsilon$$

$$\left|\frac{3}{1-x^3}\right|<\varepsilon$$

$$\frac{|3|}{|1-x^3|}<\varepsilon$$

$$\frac{3}{|1-x^3|}<\varepsilon$$

Since $x\to\infty$, we assume $x>1$ and $1-x^3<0$ to obtain:

$$\frac{3}{-(1-x^3)}<\varepsilon$$

$$\frac{3}{1-x^3}>-\varepsilon$$

$$3<-\varepsilon+\varepsilon x^3$$

$$3+\varepsilon<\varepsilon x^3$$

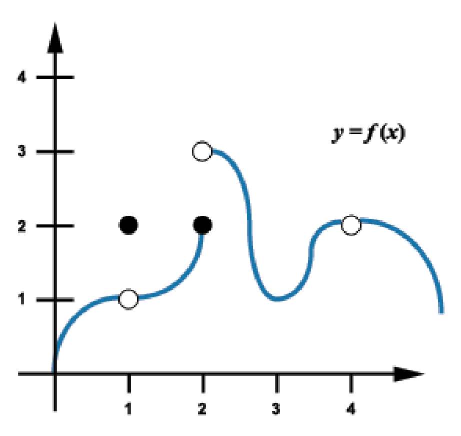
$$\sqrt[3]{\frac{3+\varepsilon}{\varepsilon}}<x$$

$$x>\sqrt[3]{\frac{3+\varepsilon}{\varepsilon}}$$

Hence, $\left|\left(\dfrac{2x^3+1}{1-x^3}\right)-(-2)\right|<\varepsilon$ whenever $x>\sqrt[3]{\dfrac{3+\varepsilon}{\varepsilon}}$; therefore, $\lim\limits_{x\to\infty}\left(\dfrac{2x^3+1}{1-x^3}\right)=-2$ by definition. Reversing the steps completes the proof.

Example Exercise 1

Consider the graph of $f(x)$ below.



A) Evaluate: $f(1)$.

B) Evaluate: $\lim\limits_{x\to 1} f(x)$.

C) Evaluate: $\lim\limits_{x\to 2^-} f(x)$.

D) Evaluate: $\lim\limits_{x\to 2^+} f(x)$.

E) Evaluate: $f(2)$.

F) Evaluate: $\lim\limits_{x\to 2} f(x)$.

G) Evaluate: $f(4)$.

H) Evaluate: $\lim\limits_{x\to 4} f(x)$.

Interpret the graph to obtain the following:

A) $f(1)=2$

B) $\lim\limits_{x\to 1} f(x)=1$

C) $\lim\limits_{x\to 2^-} f(x)=2$

D) $\lim\limits_{x\to 2^+} f(x)=3$

E) $f(2)=2$

F) $\lim\limits_{x\to 2} f(x)=D.N.E.$ because $\lim\limits_{x\to 2^-} f(x)\neq \lim\limits_{x\to 2^+} f(x)$

G) $f(4)$ is undefined

H) $\lim\limits_{x\to 4} f(x)=2$

Example Exercise 2

Find a range of possible values for the positive number δ such that $\left|\left(x^2-3\right)-1\right|<0.01$ whenever $\left|x-2\right|<\delta$.

Simplify the argument of the absolute value.

$$\begin{aligned}&\left|\left(x^2-3\right)-1\right|<.01\\&\left|x^2-4\right|<.01\end{aligned}$$

Rewrite the inequality without absolute value and simplify

$$\begin{aligned}&-0.01<x^2-4<0.01\\&-0.01+4<x^2<0.01+4\\&3.99<x^2<4.01\\&\sqrt{3.99}<x<\sqrt{4.01}\end{aligned}$$

Subtract two from each expression.

$$\begin{aligned}&\sqrt{3.99}<x<\sqrt{4.01}\\&\sqrt{3.99}-2<x-2<\sqrt{4.01}-2\end{aligned}$$

Factor out a negative one on the left-most side of the inequality.

$$-1\left(2-\sqrt{3.99}\right)<x-2<\sqrt{4.01}-2$$

Note that $2-\sqrt{3.99}>\sqrt{4.01}-2\therefore-1\left(2-\sqrt{3.99}\right)>-1\left(\sqrt{4.01}-2\right)$. Substitute $-1\left(\sqrt{4.01}-2\right)$ for the left end of the inequality.

$$-1\left(\sqrt{4.01}-2\right)<x-2<\sqrt{4.01}-2$$

Express the inequality using an absolute value expression.

$$\left|x-2\right|<\sqrt{4.01}-2$$

Substitute δ for $\left|x-2\right|$.

$$\delta<\sqrt{4.01}-2$$

Example Exercise 3

Use the strict definition of a limit to verify $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)=6$

Show that for any given positive number ε there exists a corresponding positive number δ such that $\left|\left(\frac{x^2-9}{x-3}\right)-6\right|<\varepsilon$ whenever $|x-3|<\delta$.

$$\left|\left(\frac{x^2-9}{x-3}\right)-6\right|<\varepsilon$$

$$\left|\frac{(x+3)\,\cancel{(x-3)}}{\cancel{x-3}}-6\right|<\varepsilon$$

$$|x+3-6|<\varepsilon$$

$$|x-3|<\varepsilon$$

Hence, we see that if $\delta=\varepsilon$ then $\left|\left(\frac{x^2-9}{x-3}\right)-6\right|<\varepsilon$ whenever $|x-3|<\delta$ (where $\delta=\varepsilon$.)

Example Exercise 4

Use the strict definition of a limit at infinity to verify that $\lim_{x\to\infty}\left[e^x\right]=0$.

Show that for any given positive number ε there exists a corresponding positive number N such that $\left|e^x-0\right|<\varepsilon$ whenever $x>N$.

$$\left|e^x-0\right|<\varepsilon$$

$$\left|e^x\right|<\varepsilon$$

$$e^x<\varepsilon$$

$$x<\ln\varepsilon$$

Hence, if $x\geq\ln\varepsilon$, then $\left|e^x-0\right|<\varepsilon$. Thus, $\lim_{x\to\infty}\left[e^x\right]=0$ by definition.

Application Exercise

The Heaviside function denoted $H(t)$ describes an electric current initiated at time $t = 0$ and is named for the English polymath and electrical engineer Oliver Heaviside (1850-1925).

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.5 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Graph the Heaviside function and evaluate (if possible) the three limits below.

$$i)\ \lim_{t \to 0^-} \left[H(t) \right]$$

$$ii)\ \lim_{t \to 0^+} \left[H(t) \right]$$

$$iii)\ \lim_{t \to 0} \left[H(t) \right]$$