Practice Problems for Final Exam

#1 Find the exact value of the expressions below without using a calculator.

A. $\cos\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}\right)$

B. $\dfrac{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}{\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)}$

#2 Identify the amplitude, period, and phase shift of the function below.

$$y = 8\sin\left(\frac{\pi}{3}x - \frac{\pi^2}{6}\right)$$

#3 Find the period and the range of the function below.

$$y = -3\csc(2x) - 4$$

#4 Discuss why we restrict the domain of $\tan x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ in order to define $\tan^{-1}(x)$. Then, discuss the difference between $\sin^{-1}(x)$ and $\csc(x)$.

#5 In meteorology, the vertical distance from the ground to the base of the clouds is called the *ceiling*. To measure the ceiling, weather watchers directed a spotlight vertically overhead then measured the angle of elevation from a point 200 meters away from the spotlight to the point where the beam of the spotlight hits the cloud cover as 71.5º.

#6 Prove both identities below.

A. $\dfrac{\tan(x)}{\sec(x)} = \sin(x)$

B. $\sin(x)\cot(x) = \cos(x)$

#7 Use identities to simplify the expressions below without a calculator.

A. $2\cos^2(22.5°) - 1$

B. $2\sin\left(-\frac{\pi}{12}\right)\cos\left(-\frac{\pi}{12}\right)$

#8 Prove both identities below.

A. $(1+\cot\theta)^2 - 2\cot\theta = \dfrac{1}{(1-\cos\theta)(1+\cos\theta)}$

B. $\cos\left(x - \frac{\pi}{2}\right) = \cos x \tan x.$

#9 Prove both identities below.

A. $\frac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

B. $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

#10 Solve $\triangle ABC$ with an interior angle $\beta = 10.5°$ at vertex B and side lengths $a = 6.8$ and $c = 2.4$.

#11 Find the magnitude and direction of $\mathbf{b} = \langle \sqrt{3}, 1 \rangle$.

#12 Let $z_1 = -3 + 3i$, $z_2 = -2 - 2i$, and $z_3 = -8 + 8\sqrt{3} \cdot i$.

A. Calculate $z_1 \cdot z_2$.

B. Calculate $\sqrt[4]{z_3}$

#13 Recall the formulas for converting polar to rectangular and vice versa.

#14 Determine the standard form of the conic below. Identify any foci.

$$9x^2 - 18x + 4y^2 + 24y = 11$$

#15 Evaluate $\sum_{k=1}^{\infty} \left[10 \left(\frac{2}{5} \right)^{k-1} \right]$.

#16 Find a single rectangular equation to describe the static graph generated by the parametric equations below.

$$x = 7\sin t$$
$$y = 7\cos t$$

#17 Find all the solutions to $\sin x = \sqrt{2}/2$.

#18 Suppose $0 \le \theta \le 2\pi$. Find the values of θ ∍ $2\sin^2\left(\frac{\theta}{2}\right) = \cos(\theta)$.

SOLUTIONS

#1 A) $\cos\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}$

#1B) $\frac{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}{\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)}=\frac{\frac{\pi}{6}}{\frac{\pi}{4}}=\frac{\pi}{6}\div\frac{\pi}{4}=\frac{\pi}{6}\cdot\frac{4}{\pi}=\frac{4}{6}=\frac{2}{3}$

#2 If $y=8\sin\left(\frac{\pi}{3}x-\frac{\pi^2}{6}\right)$, then $y=8\sin\left[\frac{\pi}{3}\left(x-\frac{\pi}{2}\right)\right]$. Hence, the amplitude is 8; the period is 6 as calculated below, and the phase shift is $\pi/2$ units to the right.

$$\text{period}=\frac{2\pi}{\pi/3}=2\pi\div\frac{\pi}{3}=2\pi\cdot\frac{3}{\pi}=6$$

#3 The period is π as calculated below. The range of $\csc x$ is $(-\infty,-1)\cup(1,\infty)$. Dilating $\csc x$ by -3, changes the range to $(-\infty,-3)\cup(3,\infty)$. Translating $-3\csc x$ down 4 units changes the range to $(-\infty,-7)\cup(-1,\infty)$.

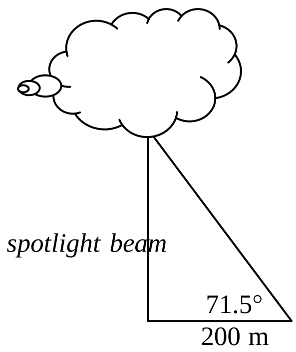
$$\text{period}=\frac{2\pi}{2}=\pi$$

#4 The graph of tangent fails the horizontal line test, meaning tangent is not a one-to-one function. Interchanging all the x and y-coordinates will create a relation where some *x*-values correspond to more than one y-value, which means that the interchange will not be a function. For instance, tangent passes through both of the points, $(0,0)$ and $(\pi,0)$. Interchanging these coordinates gives $(0,0)$ and $(0,\pi)$, but this relation has two *y*-values for the one *x*-value of zero. To alleviate this problem, we simply restrict the domain of tangent to the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$. Over this interval, the graph of tangent passes the horizontal line test.

The difference between $\sin^{-1}(x)$ and $\csc(x)$ lies in the fact that $\sin^{-1}(x)$ is a function that interchanges domain and range values of $\sin(x)$ over $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ while $\csc(x)$ simply reciprocates all the range values of $\sin(x)$. Specifically, consider the case where $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$. We say $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ and $\csc\left(\frac{\pi}{6}\right)=\frac{1}{2}$.

#5 Note that 71.5º is not a convenient value, so Mr. Simpson might tolerate the use of a calculator.

$$\tan\left(71.5°\right) = \frac{ceiling}{200 \text{ m}}$$
$$200 \text{ m} \times 2.98868 \approx ceiling$$
$$597.7 \text{ m} \approx ceiling$$



#6A) Apply the Fundamental Identity for Tangent and Reciprocal Identity for Secant as below.

$$\frac{\tan(x)}{\sec(x)} = \tan(x) \div \sec(x) = \frac{\sin(x)}{\cancel{\cos(x)}} \cdot \cancel{\cos(x)} = \sin(x) \quad ■$$

#6B) Apply the Fundamental Identity of Cotangent as below.

$$\sin(x)\cot(x) = \cancel{\sin(x)}\,\frac{\cos(x)}{\cancel{\sin(x)}} = \cos(x) \quad ■$$

#7A & B) Apply the double angle identity as below.

$$2\cos^2\left(22.5°\right) - 1 = \cos\left(2 \times 22.5°\right) = \cos\left(45°\right) = \frac{\sqrt{2}}{2}$$
$$2\sin\left(-\frac{\pi}{12}\right)\cos\left(-\frac{\pi}{12}\right) = \sin\left(2 \times -\frac{\pi}{12}\right) = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

#8A) Verify the identity, $\left(1+\cot\theta\right)^2 - 2\cot\theta = \frac{1}{\left(1-\cos\theta\right)\left(1+\cos\theta\right)}$. Squaring the binomial obtains the following.

$$1 + 2\cot\theta + \cot^2\theta - 2\cot\theta = \frac{1}{\left(1-\cos\theta\right)\left(1+\cos\theta\right)}$$

Adding like terms, the left-hand simplifies as below.

$$1 + \cot^2\theta = \frac{1}{\left(1-\cos\theta\right)\left(1+\cos\theta\right)}$$

The Pythagorean Identity, gives us the following.

$$\csc^2\theta = \frac{1}{\left(1-\cos\theta\right)\left(1+\cos\theta\right)}$$

The Reciprocal Identity changes the left-hand side as below.

$$\frac{1}{\sin^2\theta} = \frac{1}{(1-\cos\theta)(1+\cos\theta)}$$

The Pythagorean Identity then gives us the following.

$$\frac{1}{1-\cos^2\theta} = \frac{1}{(1-\cos\theta)(1+\cos\theta)}$$

Finally, using the property $a^2 - b^2 = (a-b)(a+b)$, we obtain the identity.

$$\frac{1}{(1-\cos\theta)(1+\cos\theta)} = \frac{1}{(1-\cos\theta)(1+\cos\theta)} \qquad \blacksquare$$

#8B) Verify the identity, $\cos\left(x - \frac{\pi}{2}\right) = \cos x \tan x$. By the Difference Identity of Cosine, we have:

$$\cos(x)\cos\left(\frac{\pi}{2}\right) + \sin(x)\sin\left(\frac{\pi}{2}\right) = \cos x \tan x.$$

Evaluating $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$ yields the following.

$$\cos(x)\cdot 0 + \sin(x)\cdot 1 = \cos x \tan x$$
$$\sin(x) = \cos x \tan x$$

Multiplying by a fortuitous form of “1” yields the next step.

$$\frac{\cos x}{\cos x}\cdot \sin x = \cos x \tan x$$
$$\cos x \cdot \frac{\sin x}{\cos x} = \cos x \tan x$$

The Fundamental Identity completes the verification.

$$\cos x \cdot \tan x = \cos x \tan x \qquad \blacksquare$$

#9A) Verify the identity, $\frac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$. Note, $4t = 2 \cdot 2t$. By substitution, we have:

$$\frac{\sin(2 \cdot 2t)}{4} = \cos^3 t \sin t - \sin^3 t \cos t.$$

By the Double-Angle Identity, we obtain the following on the left-hand side of the identity.

$$\frac{2\sin(2t)\cos(2t)}{4} = \cos^3 t \sin t - \sin^3 t \cos t$$
$$\frac{\sin(2t)\cos(2t)}{2} = \cos^3 t \sin t - \sin^3 t \cos t$$

By the Double-Angle Identities, we change the left-hand side as below.

$$\frac{2\sin t \cdot \cos t\left(\cos^2 t - \sin^2 t\right)}{2} = \cos^3 t \sin t - \sin^3 t \cos t$$
$$\sin t \cdot \cos t\left(\cos^2 t - \sin^2 t\right) = \cos^3 t \sin t - \sin^3 t \cos t$$

Distribution yields the identity.

$$\cos^3 t \cdot \sin t - \sin^3 t \cdot \cos t = \cos^3 t \sin t - \sin^3 t \cos t \qquad \blacksquare$$

#9B) Verify the identity, $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$. Expanding the binomial, we change the left-hand side as below.

$$\sin^2\alpha - 2\cos\alpha \cdot \sin\alpha + \cos^2\alpha = 1 - \sin 2\alpha$$

Apply the commutative property, to obtain:

$$\sin^2\alpha + \cos^2\alpha - 2\sin\alpha \cdot \cos\alpha = 1 - \sin 2\alpha .$$

Apply the Pythagorean Identity, to acquire:

$$1 - 2\sin\alpha \cdot \cos\alpha = 1 - \sin 2\alpha .$$

Thus, by the Double-Angle Identity, we complete the verification as below.

$$1 - \sin 2\alpha = 1 - \sin 2\alpha$$

$\blacksquare$

#10 Use Law of Cosines to calculate the missing side, b.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$b^2 = 6.8^2 + 2.4^2 - 2(6.8)(2.4)\cos(10.5°)$$
$$b^2 = 19.9$$
$$b \approx 4.46$$

Use Law of Sines to calculate a missing angle, keeping in mind that α is opposite the longest side of the triangle.

$$\frac{\sin(10.5°)}{4.46} = \frac{\sin(\alpha')}{6.8}$$
$$0.277848 \approx \sin(\alpha')$$
$$\sin^{-1}(0.277848) \approx \alpha'$$
$$16.1° \approx \alpha'$$
$$\alpha \approx 180° - 16.1° = 163.9°$$

The angle opposite side *a* must be obtuse from the construction of the triangle from the given information. The last angle is trivial.

#11 Denote magnitude of $\mathbf{b}$ as $|\mathbf{b}|$ and calculate as below.

$$|\mathbf{b}| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$
$$|\mathbf{b}| = \sqrt{3+1}$$
$$|\mathbf{b}| = 2$$

Denote the direction of $\mathbf{b}$ as θ and use the definition of the trigonometric functions to calculate.

$$\sin(\theta) = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}$$

#12A) Calculate directly as below or convert to trigonometric form and use the formula $r_1 r_2\left(\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right)$.

$$
\begin{aligned}
z_1\cdot z_2 &= \left(-3+3i\right)\left(-2-2i\right)\\
z_1\cdot z_2 &= -3\left(1-i\right)\left(-2-2i\right)\\
z_1\cdot z_2 &= 6\left(1-i\right)\left(1+i\right)\\
z_1\cdot z_2 &= 6\left(1-i^2\right)\\
z_1\cdot z_2 &= 6\left(1+1\right)\\
z_1\cdot z_2 &= 12
\end{aligned}
$$

#12B) Use the formula $\sqrt[n]{z}=r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+360^\circ k}{n}\right)+i\sin\left(\frac{\theta+360^\circ k}{n}\right)\right]$ for $z=r\left(\cos\theta+i\sin\theta\right)$ as below.

If $z_3=-8+8\sqrt{3}\cdot i$, then $r=16$ and $\theta=120^\circ$.

$$\sqrt[4]{z}=16^{\frac{1}{4}}\left[\cos\left(\frac{120^\circ+360^\circ(0)}{4}\right)+i\sin\left(\frac{120^\circ+360^\circ(0)}{4}\right)\right]=2\left[\frac{\sqrt{3}}{2}+\frac{1}{2}i\right]=\sqrt{3}+i$$

$$\sqrt[4]{z}=16^{\frac{1}{4}}\left[\cos\left(\frac{120^\circ+360^\circ(1)}{4}\right)+i\sin\left(\frac{120^\circ+360^\circ(1)}{4}\right)\right]=2\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]=-1+\sqrt{3}\cdot i$$

$$\sqrt[4]{z}=16^{\frac{1}{4}}\left[\cos\left(\frac{120^\circ+360^\circ(2)}{4}\right)+i\sin\left(\frac{120^\circ+360^\circ(2)}{4}\right)\right]=2\left[-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right]=-\sqrt{3}-i$$

$$\sqrt[4]{z}=16^{\frac{1}{4}}\left[\cos\left(\frac{120^\circ+360^\circ(3)}{4}\right)+i\sin\left(\frac{120^\circ+360^\circ(3)}{4}\right)\right]=2\left[\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]=1-\sqrt{3}\cdot i$$

#13 Recall the following.

$$
\begin{aligned}
&\sin\left(\theta\right)=\frac{y}{r}\Rightarrow y=r\sin\left(\theta\right)\\
&\cos\left(\theta\right)=\frac{x}{r}\Rightarrow x=r\cos\left(\theta\right)\\
&x^2+y^2=r^2
\end{aligned}
$$

#14 Complete the square as follows.

$$
\begin{aligned}
&9x^2 - 18x + 4y^2 + 24y = 11\\
&9\left(x^2 - 2x\right) + 4\left(y^2 + 6y\right) = 11\\
&9\left(x^2 - 2x + \left(\frac{2}{2}\right)^2\right) + 4\left(y^2 + 6y + \left(\frac{6}{2}\right)^2\right) = 11 + 9\cdot 1 + 4\cdot 9\\
&9\left(x^2 - 2x + 1\right) + 4\left(y^2 + 6y + 9\right) = 11 + 9 + 36\\
&9(x-1)^2 + 4(y+3)^2 = 56\\
&\frac{9(x-1)^2}{56} + \frac{4(y+3)^2}{56} = \frac{56}{56}\\
&\frac{(x-1)^2}{56/9} + \frac{(y+3)^2}{14} = 1
\end{aligned}
$$

Identify the conic as an ellipse with a vertical major axis. Identify the center as $(h,k) = (1,-3)$. Calculate the distance from the center to the foci as below.

$$
\begin{aligned}
&c^2 = \sqrt{14}^2 - \left(\sqrt{56/9}\right)^2\\
&c^2 = 14 - 56/9\\
&c^2 = \frac{70}{9}\\
&c = \frac{\sqrt{70}}{3}
\end{aligned}
$$

Hence, the foci are $\left(1, -3 \pm \frac{\sqrt{70}}{3}\right)$.

#15 Note that $\frac{2}{5} < 1$. $\sum_{k=1}^{\infty}\left[10\left(\frac{2}{5}\right)^{k-1}\right] = \frac{a_1}{1-r} = \frac{10}{1-\frac{2}{5}} = \frac{10}{\frac{3}{5}} = 10\cdot\frac{5}{3} = \frac{50}{3}$.

#16 Square both sides as follows.

$$\begin{aligned} x &= 7\sin t \\ y &= 7\cos t \\ &\Downarrow \\ x^2 &= 49\sin^2 t \\ y^2 &= 49\cos^2 t \end{aligned}$$

Now, add the two equations and use the Pythagorean Identity.

$$\begin{aligned} x^2 + y^2 &= 49\sin^2 t + 49\cos^2 t \\ x^2 + y^2 &= 49\left(\sin^2 t + \cos^2 t\right) \\ x^2 + y^2 &= 49(1) \\ x^2 + y^2 &= 7^2 \end{aligned}$$

#17 Find all the solutions to $\sin x = \sqrt{2}/2$. Note that $\sin(x)$ equals $\sqrt{2}/2$ when $x = 45°$ and when $x = 135°$. To find all the solutions, worry about all the co-terminal angles that occur with full rotations from either of these angles.   Hence, $x = 45° + 360°k$ or $x = 135° + 360°k$ where $k \in \mathbb{Z}$.

#18 We solve the conditional identity as below.

$$2\sin^2\left(\frac{\theta}{2}\right)=\cos(\theta)$$

We can simplify the left-hand side using the Pythagorean Identity. This reduces the equation to cosine expressions only.

$$2\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)=\cos(\theta)$$

We can rewrite θ as $2\cdot\frac{\theta}{2}$ to see that the Double-Angle Identity can be applied as follows.

$$2\left(1-\cos^2\left(\frac{\theta}{2}\right)\right)=\cos\left(2\cdot\frac{\theta}{2}\right)$$

$$2-2\cos^2\left(\frac{\theta}{2}\right)=2\cos^2\left(\frac{\theta}{2}\right)-1$$

Now, we solve as below.

$$2-2\cos^2\left(\frac{\theta}{2}\right)=2\cos^2\left(\frac{\theta}{2}\right)-1$$

$$3-2\cos^2\left(\frac{\theta}{2}\right)=2\cos^2\left(\frac{\theta}{2}\right)$$

$$3=4\cos^2\left(\frac{\theta}{2}\right)$$

$$\cos^2\left(\frac{\theta}{2}\right)=\frac{3}{4}$$

$$\cos\left(\frac{\theta}{2}\right)=\pm\frac{\sqrt{3}}{2}$$

$$\frac{\theta}{2}=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6},\ldots$$

Multiplying by 2 on both sides, we find the solutions, but keep in mind the restrictions, namely, $0\le\theta\le 2\pi$.

$$\theta=\frac{\pi}{3},\frac{5\pi}{3}$$