Basic Identities

A large part of studying trigonometry involves studying the numerous interrelationships among the trigonometric functions. With this lecture, we begin verifying trigonometric identities. This particular lecture will list identities already mentioned in previous lectures then demonstrate how to prove new relationships using these basic identities. We start by listing previously mentioned identities.

Fundamental Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cos\theta \neq 0$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta},\ \sin\theta \neq 0$$

Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta},\ \sin\theta \neq 0$$
$$\sin\theta = \frac{1}{\csc\theta},\ \csc\theta \neq 0$$
$$\sec\theta = \frac{1}{\cos\theta},\ \cos\theta \neq 0$$
$$\cos\theta = \frac{1}{\sec\theta},\ \sec\theta \neq 0$$
$$\cot\theta = \frac{1}{\tan\theta},\ \tan\theta \neq 0$$
$$\tan\theta = \frac{1}{\cot\theta},\ \cot\theta \neq 0$$

Odd Identities:

$$-\sin(x) = \sin(-x)$$
$$-\csc(x) = \csc(-x)$$
$$-\tan(x) = \tan(-x)$$
$$-\cot(x) = \cot(-x)$$

Even Identities:

$$\cos(x) = \cos(-x)$$
$$\sec(x) = \sec(-x)$$

We have also mentioned the Pythagorean Identity, namely, $\sin^2\theta+\cos^2\theta=1$. There are two more identities commonly referred to as Pythagorean Identities. The box below gives all three.

Pythagorean Identities:

$$\sin^2\theta+\cos^2\theta=1$$
$$1+\tan^2\theta=\sec^2\theta$$
$$1+\cot^2\theta=\csc^2\theta$$

Deriving the second two Pythagorean identities is quite simple based on the first. For instance, consider the first Pythagorean identity. If we divide through by $\cos^2\theta$ we arrive at the second Pythagorean identity as shown below.

$$\sin^2\theta+\cos^2\theta=1$$
$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

Of course, dividing by a quantity introduces the difficulty of worrying about whether or not the divisor-quantity equals zero, but in this case tangent and secant are not defined if $\cos\theta\neq 0$, so we simply divide for all values of θ except where $\cos\theta=0$. To conclude, we employ the fundamental and reciprocal identities as below.

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$
$$\tan^2\theta+1=\sec^2\theta$$

We leave the proof for $1+\cot^2\theta=\csc^2\theta$ to the reader.

The main point of this lecture is to demonstrate how to use the basic identities to verify other identities. For example, we will show that $\csc\theta-\cos\theta\cot\theta=\sin\theta$. First, we apply the reciprocal identity for cosecant as below.

$$\csc\theta-\cos\theta\cot\theta=\sin\theta$$
$$\frac{1}{\sin\theta}-\cos\theta\cot\theta=\sin\theta$$

Second, we apply the fundamental identity for cotangent.

$$\frac{1}{\sin\theta}-\cos\theta\cot\theta=\sin\theta$$
$$\frac{1}{\sin\theta}-\cos\theta\frac{\cos\theta}{\sin\theta}=\sin\theta$$

Next, we add the resulting fractions as follows.

$$\frac{1}{\sin\theta}-\cos\theta\frac{\cos\theta}{\sin\theta}=\sin\theta$$
$$\frac{1}{\sin\theta}-\frac{\cos^2\theta}{\sin\theta}=\sin\theta$$
$$\frac{1-\cos^2\theta}{\sin\theta}=\sin\theta$$

Noting that $\sin^2\theta+\cos^2\theta=1$ implies $\sin^2\theta=1-\cos^2\theta$, we conclude as below.

$$\frac{1-\cos^2\theta}{\sin\theta}=\sin\theta$$
$$\frac{\sin^2\theta}{\sin\theta}=\sin\theta$$
$$\frac{\sin\theta\cdot\cancel{\sin\theta}}{\cancel{\sin\theta}}=\sin\theta$$

Since all the steps performed on the left can be reversed, we conclude $\csc\theta-\cos\theta\cot\theta=\sin\theta$.

For a second example, we will verify that $(1-\sin x)(1+\sin x)=\frac{1}{1+\tan^2 x}$. We start by using the algebraic identity, $(a+b)(a-b)=a^2-b^2$ as below.

$$(1-\sin x)(1+\sin x)=\frac{1}{1+\tan^2 x}$$
$$1-\sin^2 x=\frac{1}{1+\tan^2 x}$$

Since the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$ implies $\sin^2\theta=1-\cos^2\theta$, we can substitute next as below.

$$1-\sin^2 x=\frac{1}{1+\tan^2 x}$$
$$\cos^2 x=\frac{1}{1+\tan^2 x}$$

Applying the reciprocal identity followed by the Pythagorean identity $1+\tan^2\theta=\sec^2\theta$, we conclude as follows.

$$\cos^2 x = \frac{1}{1+\tan^2 x}$$
$$\frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x}$$
$$\frac{1}{1+\tan^2 x} = \frac{1}{1+\tan^2 x}$$

Verifying a non-identity is a simple matter of finding a counter-example. For instance, consider the equation $2\sin\theta\cos\theta - \sin\theta = 0$. This equation holds if $\theta = 0$, but to show that the equation is not an identity, we simply let $\theta = \pi/2$ as follows.

$$2\sin\theta\cos\theta - \sin\theta \stackrel{?}{=} 0$$
$$2\sin(\pi/2)\cos(\pi/2) - \sin(\pi/2) \stackrel{?}{=} 0$$
$$2\cdot 1\cdot 0 - 1 \stackrel{?}{=} 0$$
$$-1 \neq 0$$

As implied above, an identity can usually be verified several ways. However, the suggestions below are helpful.

Suggestions for Verifying Identities

1. Work with only one side of the equation at a time. It is usually easier to begin with the more complicated side and then simplifying using the remaining suggestions.

2. Make substitutions using known identities. Often it is helpful to rewrite one side in terms of sine or cosine.

3. Perform indicated algebraic operations such as adding or subtracting rational expressions or multiplying polynomials.

4. Consider reversing algebraic operations such as factoring polynomials and decomposing rational expressions.

5. Keep checking the result against the other side of the identity.

Example Exercise 1

Verify $\dfrac{-1}{\tan\theta - \sec\theta} = \dfrac{1+\sin\theta}{\cos\theta}$.

Multiply the left side by a propitious choice of "1" to obtain the following.

$$\frac{-1}{\tan\theta - \sec\theta} \cdot \frac{\tan\theta + \sec\theta}{\tan\theta + \sec\theta} = \frac{1+\sin\theta}{\cos\theta}$$

$$\frac{-1(\tan\theta + \sec\theta)}{\tan^2\theta - \sec^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$

By the Pythagorean Identity, we have $\tan^2\theta = \sec^2\theta - 1$; hence, we have

$$\frac{-1(\tan\theta + \sec\theta)}{\sec^2\theta - 1 - \sec^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$

$$\frac{-1(\tan\theta + \sec\theta)}{\sec^2\theta - \sec^2\theta - 1} = \frac{1+\sin\theta}{\cos\theta}$$

$$\frac{\cancel{-1}(\tan\theta + \sec\theta)}{\cancel{-1}} = \frac{1+\sin\theta}{\cos\theta}$$

$$\tan\theta + \sec\theta = \frac{1+\sin\theta}{\cos\theta}.$$

Next, we substitute a Fundamental Identity and a Reciprocal Identity as below.

$$\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

Adding the fractions yields the identity.

$$\frac{\sin\theta + 1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

Example Exercise 2

Verify $\cot(x)\cdot\sin(x)-\cos^2(x)\cdot\sec(x)=0.$

By Fundamental Identity, we have the following.

$$\frac{\cos(x)}{\sin(x)}\cdot\sin(x)-\cos^2(x)\cdot\sec(x)=0$$

Next, we reduce as below.

$$\frac{\cos(x)}{\cancel{\sin(x)}}\cdot\cancel{\sin(x)}-\cos^2(x)\cdot\sec(x)=0$$

$$\cos(x)-\cos^2(x)\cdot\sec(x)=0$$

By Reciprocal Identity and reduction, we conclude as below.

$$\cos(x)-\cos^2(x)\cdot\frac{1}{\cos(x)}=0$$

$$\cos(x)-\cos^{\cancel{2}}(x)\cdot\frac{1}{\cancel{\cos(x)}}=0$$

$$\cos(x)-\cos(x)=0$$

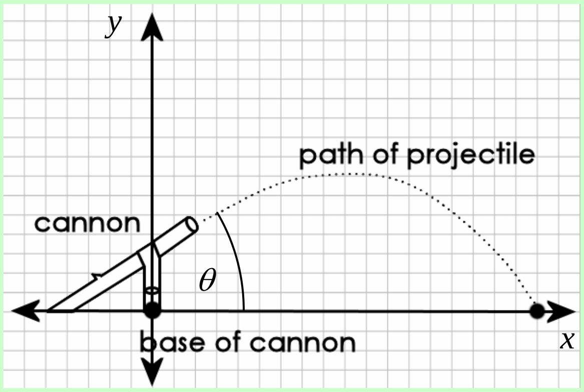
$$0=0$$

Suggested Homework

Section 6.2: #55-87 odd

Application Exercise

A projectile is fired with initial velocity of v_0. The projectile can be pictured as being fired from the origin into the first quadrant, making an angle θ with the positive x-axis as shown in the figure below.



If there is no significant air resistance, then at time t the coordinates of the projectile are $x = v_0 t\cos\theta$ and $y = -16t^2 + v_0 t\sin\theta$. Show that $y = -\frac{16}{v_0^2}\sec^2\theta x^2 + x\tan\theta$.