

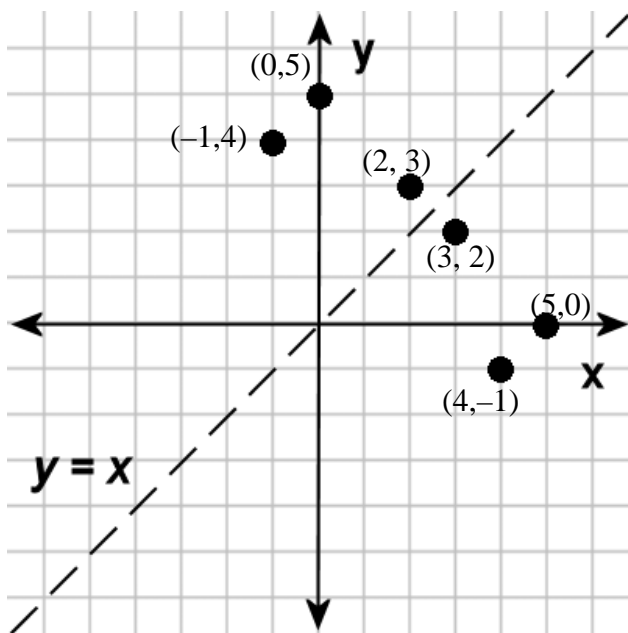
Inverse Trigonometric Functions

Any one-to-one function has an inverse. A function is one-to-one if no two ordered pairs in the function have the same second component. In other words, one-to-one functions do not have any repeated values. Consequently, $y = \sin x$ is not one-to-one. A quick test for one-to-one correspondence is the horizontal-line test. If all the horizontal lines of the Cartesian plane intersect the function only once, the function is one-to-one and has an inverse. A quick glance at the trigonometric functions shows that all six fail the horizontal line test. Hence, the trigonometric functions are only invertible when we restrict their domain over an interval so that the resulting graph will pass the horizontal-line test.

If a function is a one-to-one function over a given interval, then its inverse is the function formed by reversing all the ordered pairs.

Suppose $f(x)$ is a one-to-one function over a given interval I . Then, reversing all the ordered pairs of $f(x)$ creates a function we call the inverse of $f(x)$, denoted $f^{-1}(x)$.

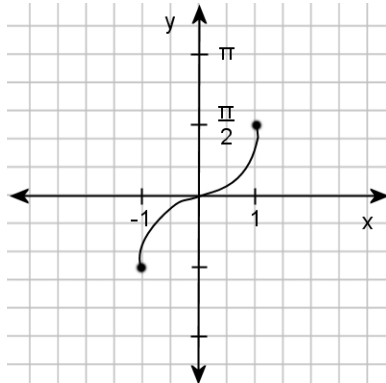
In other words, if a function is a one-to-one function, then its inverse is the function formed by reversing all the ordered pairs. For example, for the function $f : \{(-1,4), (2,3), (5,0)\}$, the inverse is $f^{-1} : \{(4,-1), (3,2), (0,5)\}$. Note the importance of the original function being one-to-one: if the original function is not one-to-one, then reversing the ordered pairs would yield a relation that is not a function. Graphically, the points of f^{-1} are reflections of the points of f across the line $y = x$, as shown below.



Every one-to-one function has an inverse, and the graph of the inverse of any one-to-one function f can be found by reflecting the graph of f across the line $y = x$.

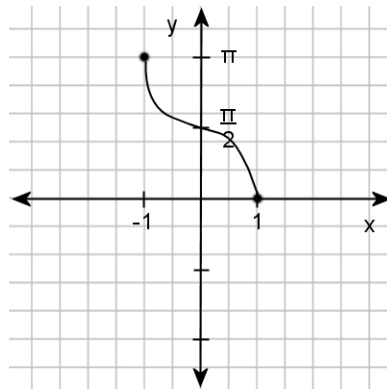
We can define inverse functions for all six trigonometric functions by restricting their domain. However, we will concern ourselves only with inverses of sine, cosine, and tangent. We denote the inverse of sine as $\sin^{-1}(x)$ or $\arcsin(x)$.

$$y = \sin^{-1}(x) \text{ or } y = \arcsin x \text{ means } \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$



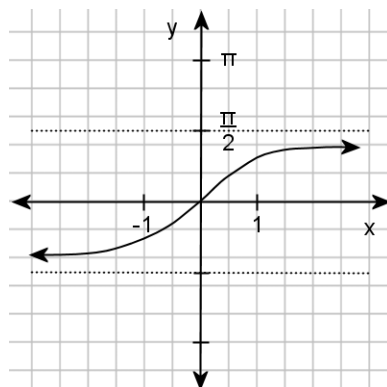
Similarly, we denote the inverse of cosine as $\cos^{-1}(x)$ or $\arccos(x)$.

$$y = \cos^{-1}(x) \text{ or } y = \arccos x \text{ means } \cos y = x \text{ and } 0 \leq y \leq \pi.$$



Likewise, we denote the inverse of tangent as $\tan^{-1}(x)$ or $\arctan(x)$.

$$y = \tan^{-1}(x) \text{ or } y = \arctan x \text{ means } \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$



In the definitions above, the range restrictions result from arbitrary restrictions on the domain of sine, cosine, and tangent respectively. It is usual to restrict the domain of sine to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that the function will be one-to-one and have an inverse. Other restrictions could be chosen. The usual restriction for cosine is $[0, \pi]$ while the usual restriction for tangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Recall that we often think of the input values of a trigonometric function as angle measures and their outputs as the ratios of side lengths (although we went through a great deal of trouble to realize the inputs can be any real number for sine and cosine). Since inverse functions reverse the ordered pairs, it is helpful to think of the inputs as ratios of side lengths and the outputs as angle measures in inverse trigonometric functions. For example, consider a few ordered pairs of $y = \sin x$.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
y	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1

Since the inverse function reverses the ordered pairs, we have the table below for $y = \arcsin x$.

x	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
y	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$

Note how the convenient x -values for $y = \sin x$ are common angle measures, but for the inverse function, the common angle measures are y -values.

Example Exercise 1

Find the exact value of $\arcsin(\sqrt{3}/2)$.

We start by restricting the domain of sine so that it is a one-to-one function. If we restrict the domain of sine to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then its graph passes the horizontal line test, which means sine is one-to-one over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and has an inverse.

Now, we recall that arcsine is the function that produces the set of ordered pairs that are interchanges between x and y -values of the sine function. (Recall also that we think about the x -values of sine as angle measures despite the fact that the inputs are really just real numbers. This means that the outputs of arcsine are numbers that we can think of as angle measures.) For $\arcsin(\sqrt{3}/2)$ then, we are looking for the angle measure from $-\pi/2$ and $\pi/2$ where sine takes the value $\sqrt{3}/2$, which is $\pi/3$.

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Example Exercise 2

Find the exact value of $\cot^{-1}(1)$.

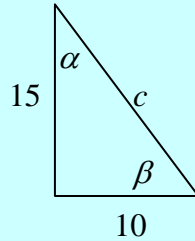
We start by restricting the domain of tangent so that it is a one-to-one function. If we restrict the domain of cotangent $[0, \pi]$, then its graph passes the horizontal line test, which means cotangent is one-to-one over $[0, \pi]$ and has an inverse.

Now, we recall that inverse cotangent is the function that produces the set of ordered pairs that are interchanges between x and y -values of the cotangent function. (Recall also that we think about the x -values of sine as angle measures despite the fact that the inputs are really just real numbers. This means that the outputs of arcsine are numbers that we can think of as angle measures.) For $\cot^{-1}(1)$ then, we are looking for the angle measure from 0 and π where cotangent equals one. Remember that cotangent equals the ratio or cosine to sine, so for cotangent to equal one, cosine and sine must be equal. These two functions equal one another at $\pi/4$. Hence, cotangent equals one at $\pi/4$. Hence, inverse cotangent equals $\pi/4$ at one.

$$\cot^{-1}(1) = \frac{\pi}{4}$$

Example Exercise 3

Solve the triangle in the figure below. In other words, find the angle measures for alpha and beta and the length for c .



We use the Pythagorean Theorem to find the hypotenuse.

$$\begin{aligned}15^2 + 10^2 &= c^2 \\225 + 100 &= c^2 \\325 &= c^2 \\\sqrt{325} &= \sqrt{c^2} \\\sqrt{13 \cdot 25} &= c \\5\sqrt{13} &= c\end{aligned}$$

Now, we turn to the right triangle theorem for the trigonometric functions, which gives us the relationship, $\sin(\alpha) = \frac{\text{opposite side length}}{\text{hypotenuse length}}$. Thus, we have $\sin(\alpha) = \frac{10}{5\sqrt{13}}$, and we solve using a table of arcsine values (i.e., our calculator) as below.

$$\begin{aligned}\sin(\alpha) &= \frac{10}{5\sqrt{13}} \\\sin(\alpha) &\approx (0.5547001962) \\\alpha &\approx \sin^{-1}(0.5547001962) \\\alpha &\approx 33.7^\circ\end{aligned}$$

We find β using the fact that the interior angles of a triangle are supplementary.

$$\begin{aligned}90^\circ + \alpha + \beta &= 180^\circ \\90^\circ + 33.7^\circ + \beta &\approx 180^\circ \\123.7^\circ + \beta &\approx 180^\circ \\\beta &\approx 180^\circ - 123.7^\circ \\\beta &\approx 56.3^\circ\end{aligned}$$

Suggested Homework

Section 5.5: #1-11 odd, #17-27 odd, #33, #41, #65, #67, #71

Section 5.6: #25-31 odd

Application Exercise

The function $x = \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t)$ gives the position x at time t for a weight in motion on a vertical spring where v_0 is the initial velocity, x_0 is the initial position, and ω is a constant relative to the particular spring.

At time $t = 0$, a 1-kilogram block is positioned so that the spring compresses to a position 1 meter below the zero position. When released from this position, the block moves upward with an initial velocity of 2 meters per second. Assume $\omega = 1$. Hence, $x = 2 \sin(t) - \cos(t)$ gives the position of the block in relation to the zero position after t seconds. When is the weight at the zero position? In other words, what are the solutions to the equation below?

$$2 \sin t - \cos t = 0$$

Hint: we can simplify the equation as follows.

$$2 \sin t - \cos t = 0$$

$$2 \sin t = \cos t$$

$$2 \frac{\sin t}{\cos t} = 1$$

$$\frac{\sin t}{\cos t} = \frac{1}{2}$$

$$\tan t = \frac{1}{2}$$