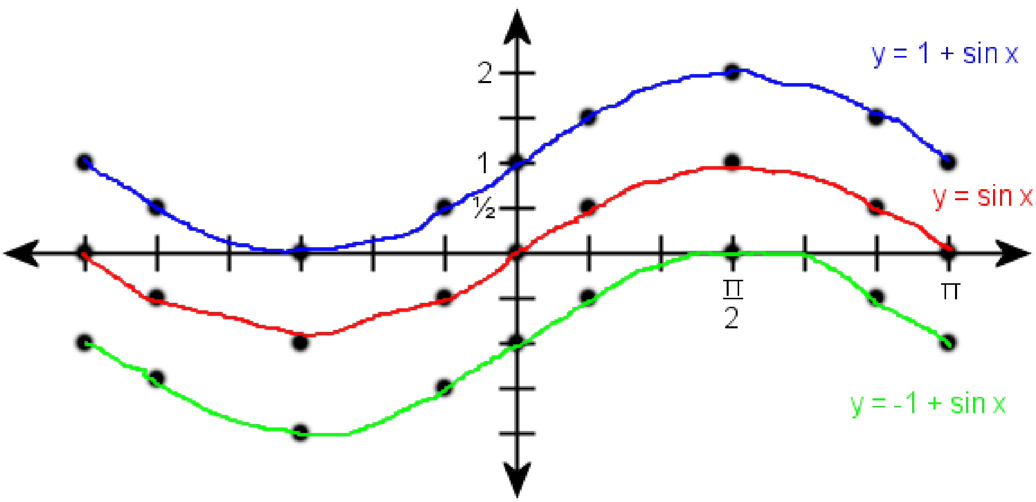
Translations of Trigonometric Functions

In the last lecture, we dilated and reflected the graphs of trigonometric functions. In this lecture, we will discuss other type of transformations called translations. While dilations change the shape of a graph, translations simply move the graph. Accordingly, we call translations shifts because they shift the graph to a new location. We will concern ourselves with two types of translations, vertical and horizontal. A vertical translation simply moves the graph up or down some set number of units.

Let $y = f(x)$, then we call $y = f(x) + C$ a *vertical translation* of $f(x)$ and C represents the number of units each point on the graph is translated vertically. If $C > 0$, then the points of $y = f(x) + C$ lie C-units *above* their corresponding points on $y = f(x)$. If $C < 0$, then the points of $y = f(x) + C$ lie $|C|$-units *below* their corresponding points on $y = f(x)$.

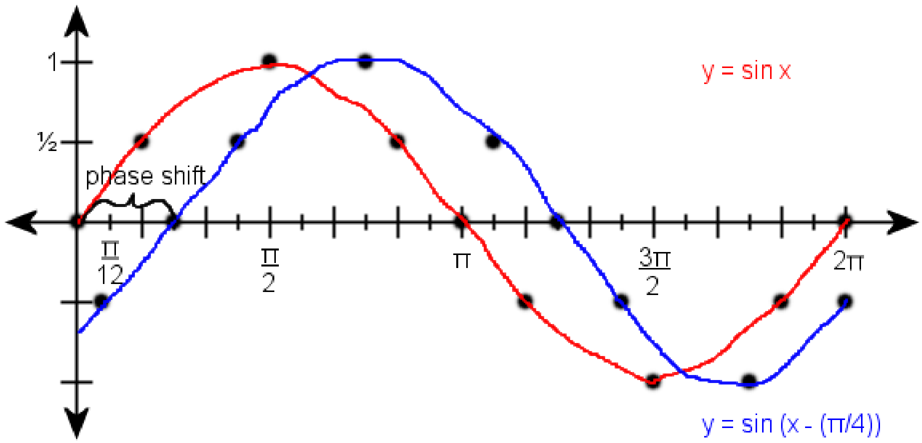
The hand-drawn graphs below show how a vertical shift affects a parent graph.



Horizontal translations shift the graph left or right. We call horizontal translations *phase shifts*.

Let $y = f(x)$ be a periodic function, then we call $y = f(x - c)$ a *horizontal translation* of $f(x)$, and we call c the *phase shift*. The phase shift represents the number of units each point on the graph is translated horizontally. If $c > 0$, then the points of $y = f(x - c)$ lie c-units *to the right* of their corresponding points on $y = f(x)$. If $c < 0$, then the points of $y = f(x - c)$ lie $|c|$-units *to the left* of their corresponding points on $y = f(x)$.

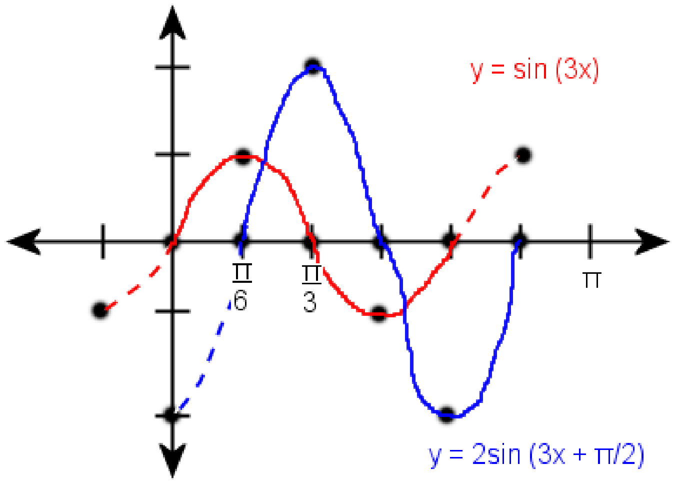
The hand-drawn graph below shows a translation to the right. The phase shift equals $\pi/4$.



Graphing any single transformation is straightforward, but generating a graph with more than one transformation is a little trickier. Consider the function $y = 2\sin(3x + \pi/2)$. We start by finding the phase shift as below.

$$y = 2\sin(3x + \pi/2)$$
$$y = 2\sin\left[3\left(x + \frac{\pi}{6}\right)\right]$$
$$y = 2\sin\left[3\left(x - \left(-\frac{\pi}{6}\right)\right)\right]$$

Normally, $y = \sin(x)$ has a fundamental cycle $[0, 2\pi]$. The dilation $y = \sin 3x$ has a least period of $2\pi/3$, so it has a fundamental cycle of $[0, 2\pi/3]$. Note that $y = \sin x$ has *x*-intercepts at $x = 0$ (the beginning point of the fundamental cycle), $x = \pi$ (the halfway point), and $x = 2\pi$ (the endpoint). Accordingly, the dilation $y = \sin 3x$ has *x*-intercepts at $x = 0$ (its beginning point), $x = \pi/3$ (its half-way point), and $x = 2\pi/3$ (its end-point). However, since $y = \sin\left[3\left(x - \frac{\pi}{6}\right)\right]$ has a phase shift equal to $\pi/6$, the *x*-intercepts each move $\pi/6$-units to the right. Hence, the *x*-intercepts of $y = \sin\left[3\left(x - \frac{\pi}{6}\right)\right]$ are $x = \pi/6$, $x = \pi/2$ (because $\pi/3 + \pi/6 = 3\pi/6 = \pi/2$), and $x = 5\pi/6$. Finally, $y = 2\sin\left[3\left(x - \frac{\pi}{6}\right)\right]$ vertically dilates $y = \sin\left[3\left(x - \frac{\pi}{6}\right)\right]$ so that the amplitude changes from 1 to 2. A vertical dilation does not affect *x*-intercepts (since zero times any number is still zero), which is why it is always convenient to plot the *x*-intercepts first.



In general, it is wise to adhere to the following procedure when generating a graph with multiple transformations. The reader will note that the procedure effectively follows the order of operations.

Let $y = f(x)$ be a trigonometric function with least period *P*. To graph $y = Af(B(x-C))+D$, we complete the following steps.

1. Sketch one cycle of $y = f(Bx)$ over its fundamental cycle, $\left[0, \frac{P}{|B|}\right]$.
2. Reflect the cycle over the y-axis if $B < 0$.
3. Translate the cycle $|C|$ units to the right if $C > 0$ or $|C|$ units to the left if $C > 0$.
4. Change the amplitude of the cycle by a factor of $|A|$.
5. If $A < 0$, reflect the curve over the *x*-axis.
6. Translate the graph $|D|$ units upward if $D > 0$ or downward if $D < 0$.

Suggested Homework

Section 5.3: #17, #19, #21, #25, #27, #65, #67
Section 5.4: #53, #63, #65, #79

Application Exercise

Many functions can be approximated to any degree of accuracy by polynomial functions called Taylor polynomials. The Taylor polynomials that approximate the sine wave function are given below.

$$y = x$$

$$y = x - \frac{x^3}{3!},$$

$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!},$$

$$\vdots$$

Demonstrate the approximation of $y = \sin x$ by the third-degree Taylor polynomial $y = x - \frac{x^3}{3!}$ by sketching both functions over the interval $[-\pi, \pi]$. Keep in mind that $3! = 3 \cdot 2 \cdot 1 = 6$.