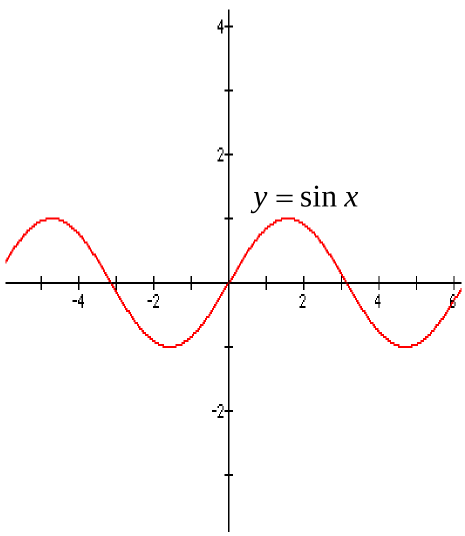
Dilations and Reflections of Trigonometric Functions

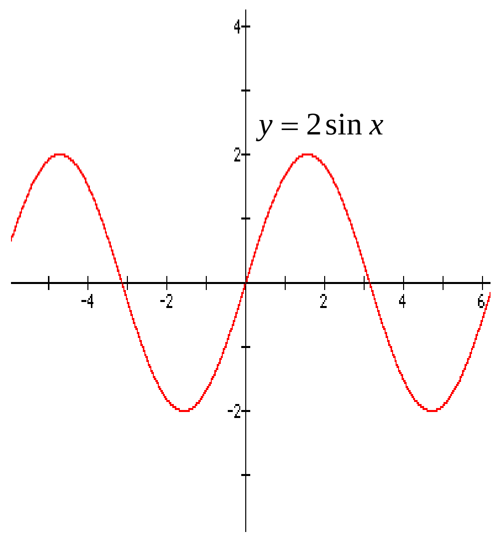
In the last lecture, we graphed the trigonometric functions. In this lecture, we will discuss dilations and reflections of these graphs. We start by defining dilation.

Let $y = f(x)$ and let $A > 0$, then we call $y = A \cdot f(x)$ a *vertical dilation* of $f(x)$ and A represents the *scale of dilation*.

Let $y = f(x)$ and let $a > 0$, then we call $y = f(ax)$ a *horizontal dilation* of $f(x)$ and a represents the *scale of dilation*.

The graphs below demonstrate the effect of a vertical dilation on a sine wave by a scale of two.

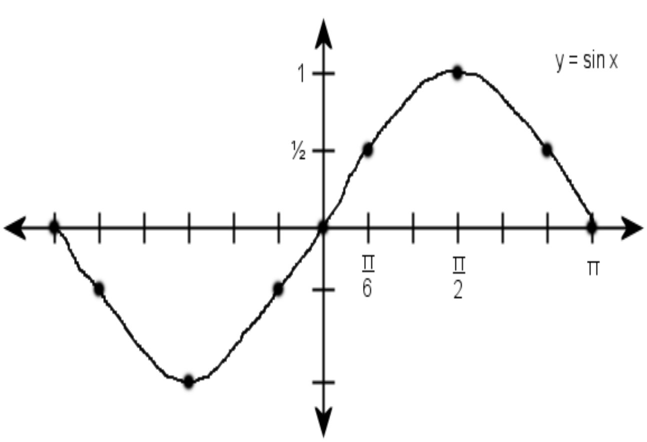


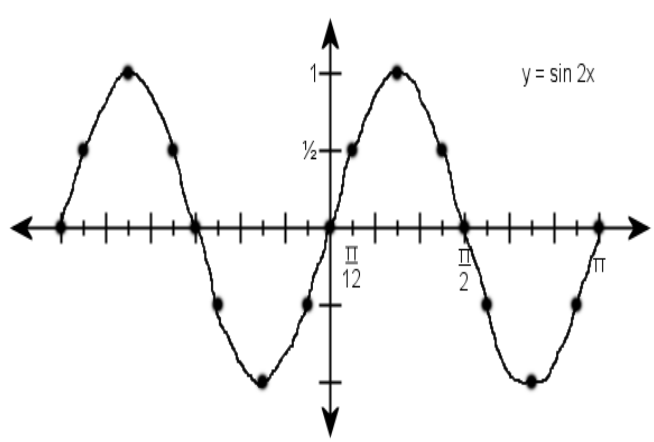


Note that the vertical dilation does not affect periodicity but has a dramatic impact on the amplitude of the wave. This demonstrates the following theorem.

The amplitude of $y = A\sin(x)$ or of $y = A\cos(x)$ equals $|A|$.

As just noted, a vertical dilation affects the amplitude of the graph. Horizontal dilations affect the periodicity of periodic functions as demonstrated by the hand-drawn graphs below.





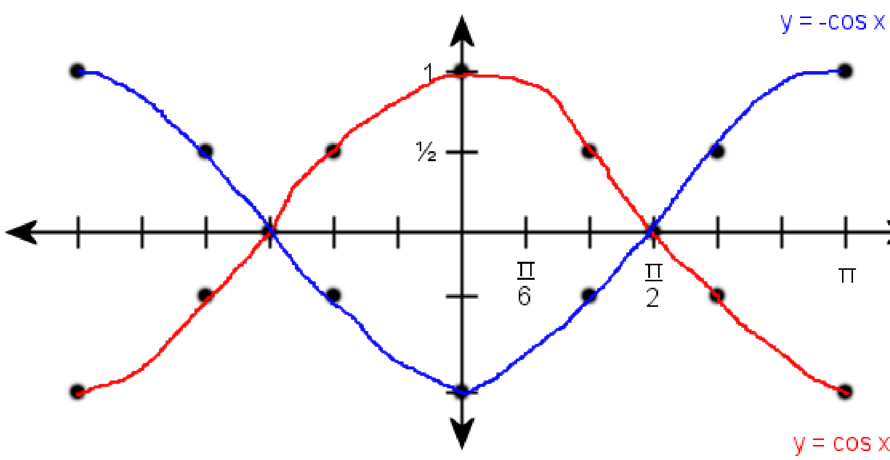
The graph of $y = \sin(2x)$ completes its full cycle twice as often as the graph of $y = \sin(x)$, which demonstrates the following theorem.

> Let P be the least period of a periodic function $y = f(x)$.
>
> Then, $\frac{P}{|a|}$ is the least period of $y = f(ax)$.

Alert readers will note that we defined the scale of dilation to be a positive number. We do that because a negative scale of dilation creates a mirror image called *reflection*.

> Let $y = f(x)$, then the graph of $y = -f(x)$ is a *reflection* of $f(x)$ over the x-axis and the graph of $y = f(-x)$ is a reflection of $f(x)$ over the y-axis.

The hand-drawn graph below demonstrates a basic reflection over the x-axis.



The topic of reflection leads us to *even* and *odd functions* defined below.

> If $f(x) = f(-x)$, then we call $f(x)$ an even function. If $-f(x) = f(-x)$, then we call $f(x)$ an odd function.

The graphs of the trigonometric functions illustrate the following identities.

> Even Identities:
>
> $$\cos(x) = \cos(-x) \text{ and } \sec(x) = \sec(-x)$$
>
> Odd Identities:
>
> $$-\sin(x) = \sin(-x) \text{ and } -\csc(x) = \csc(-x)$$
> $$-\tan(x) = \tan(-x) \text{ and } -\cot(x) = \cot(-x)$$

Suggested Homework

Section 5.3: #5, #11, #13, #15, #57, #63
Section 5.4: #51, #55, #59, #67, #71, #75

Application Exercise

Scientists use the same types of terms to describe ocean waves that describe sine waves. The *wave period* equals the time between crests. Assume that the graph of $y = \sin\left(\frac{\pi}{10}x\right)$ where x represents seconds elapsed models an ocean wave, and identify the wave period.