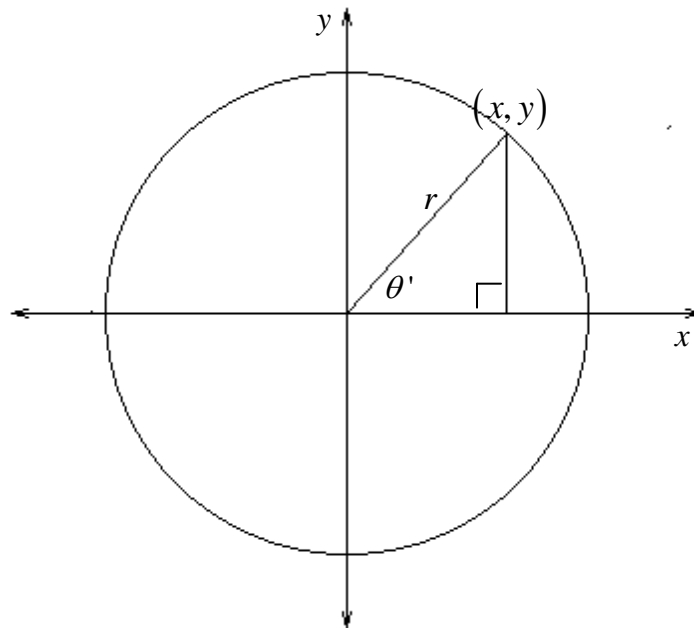


## Trigonometric Functions

In the last few lectures, we defined the trigonometric functions. In this lecture, we will discuss their graphs. We will limit this introductory discussion to the graphs without transformation. We begin with the function  $y = \sin x$ . Note the subtle change in variable designation. Previously,  $x$  and  $y$  were variables used to calculate the output of a function  $\sin \theta$  where  $\theta$  was the input. Now  $x$  represents the input and  $y$  represents the output. Or, when we let  $y = f(x)$ , then  $f(x)$  represents the output. Generally, when  $x$  represents the input of a function then  $x$  represents any real number (unless the function places some restriction on  $x$ ). This is the case with  $y = \sin x$ . The input of the sine function is the measure of an angle in standard position, but using radians or even degrees, the measure of an angle in standard position can equal any real number, rational or irrational, negative or nonnegative. Hence, the domain of sine equals  $\mathbb{R}$ , and we can assign  $x$  or  $t$  or any variable to represent domain values. Similarly, we can assign  $y$  or  $f(x)$  to represent range values.

Recall our original definition of the sine function. Consider this definition alongside the figure below.

If  $\theta$  is an angle in standard position and  $(x, y)$  is the point of intersection of the terminal side and the circle centered at the origin with radius  $r$ , then sine maps the measure of  $\theta$  to  $y/r$ . We conflate the designation of the angle with its actual measurement and denote the sine function as  $\sin(\theta) = y/r$ .

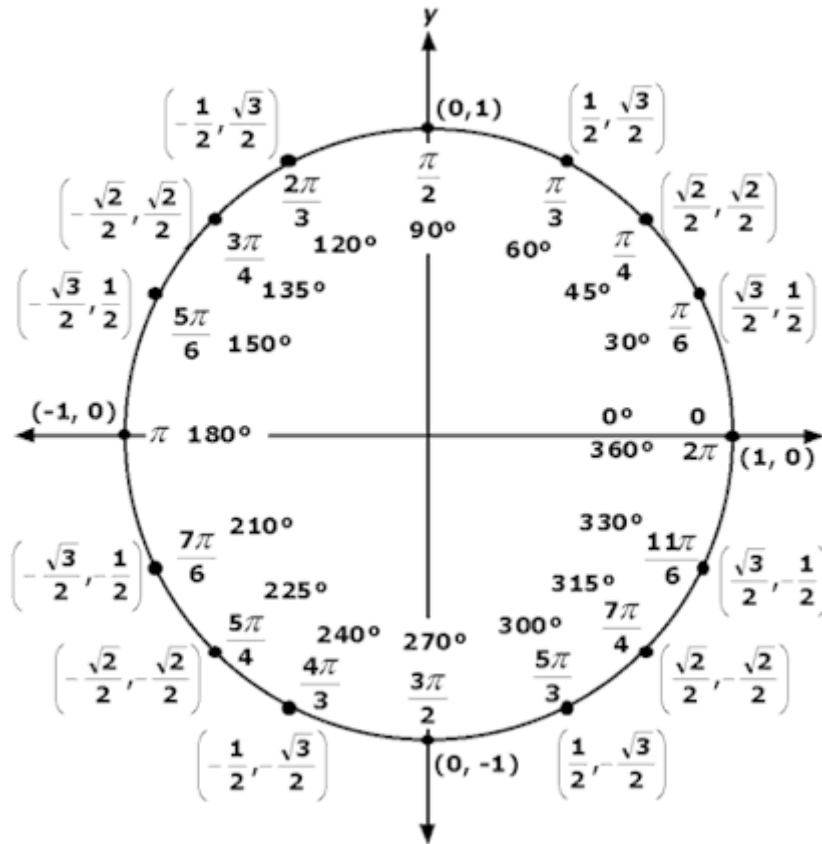


The diagram demonstrates the basic inequality  $|y| \leq r$ . When  $|y| < r$ , the output  $y/r$  equals a proper fraction. When  $|y| = r$ , then the output,  $y/r$ , equals 1. Since the radius is always positive, when  $y$  is negative  $y/r$  is negative and when  $y$  is positive  $y/r$  is positive. Hence, the

range of sine equals the interval  $[-1,1]$  while the domain equals the interval  $(-\infty,\infty)$  as discussed above.

Now recall our simplified definition using the unit circle. Consider this definition alongside the figure below.

If  $\theta$  is an angle in standard position and  $(x, y)$  is the point of intersection of the terminal side and the unit circle centered at the origin, then  $\sin(\theta) = y$ .

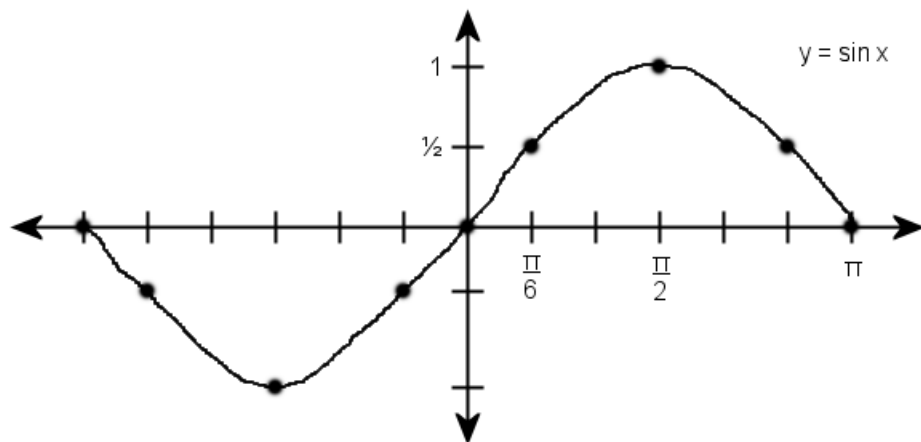


The value of  $\sin \theta$  at the multiples of  $30^\circ$  and  $45^\circ$  reveal a repetitive pattern to the function especially when the reader reflects that all these same values will be repeated as the angle measures extend beyond  $2\pi$ . For instance, sine has the same value at  $13\pi/6$  as it does at  $\pi/6$ . This repetition creates what we call a *periodic function*, one that repeats itself after a given interval called its period. We define a periodic function formally below.

A function  $f(x)$  is *periodic* if  $\exists$  a number  $P \ni f(x) = f(x+kP)$  for  $k \in \mathbb{Z}$ . The *least period* equals the smallest positive constant  $P$ .

In the case of sine, we see that  $\sin(x) = \sin(x+2k\pi)$ , so sine is a periodic function with a least period equal to  $2\pi$ . One complete least period of the sine function is called a cycle. The interval

$[0, 2\pi]$  is called the fundamental cycle for  $f(x) = \sin(x)$ . The image below shows the graph of  $f(x) = \sin(x)$  over the fundamental cycle using points from the table.



$x$	$\sin(x)$
0	0
$\pi/4$	$\sqrt{2}/2 \approx 0.7$
$\pi/2$	1
$3\pi/4$	$\sqrt{2}/2$
$\pi$	0
$5\pi/4$	$-\sqrt{2}/2 \approx -0.7$
$3\pi/2$	-1
$7\pi/4$	$-\sqrt{2}/2$
$2\pi$	0

The wave pattern repeats continuously in both directions. We call this graph or any translation or dilation of this graph a sine wave or a sinusoidal wave. We call the magnitude of the wave's oscillation its *amplitude*. We define the amplitude formally below.

Amplitude equals the maximum displacement from a zero position. Accordingly, the amplitude of a sine wave equals the absolute value of half the difference between the maximum and minimum  $y$ -values on the curve.

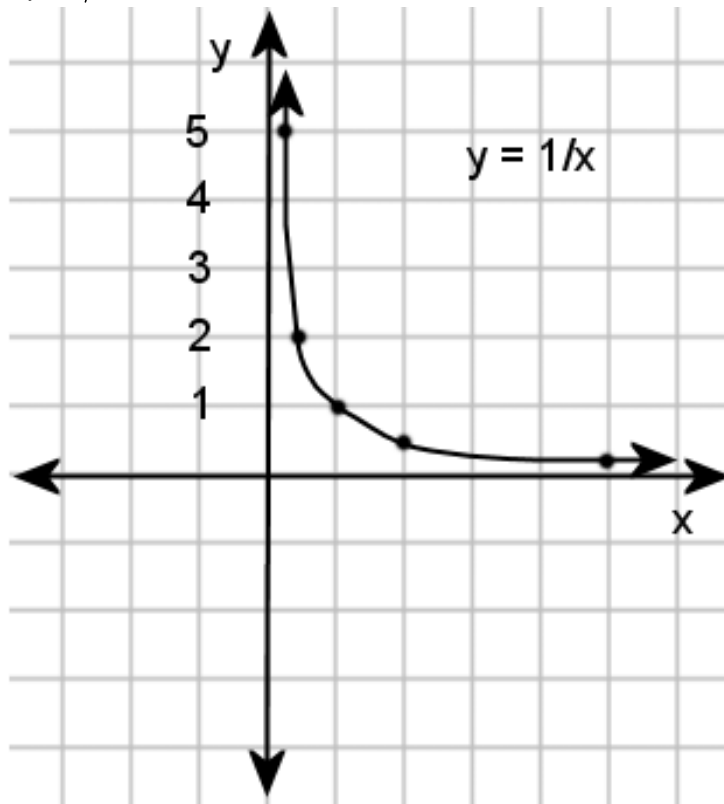
The other five trigonometric functions also represent periodic functions. Note that sine, cosine, secant, and cosecant all have a least period of  $2\pi$  while tangent and cotangent have a least period of  $\pi$ . Only sine and cosine have real number amplitudes.

We leave graphing the remaining trigonometric functions as an exercise, but before we do, we comment on a characteristic of some functions that aid in graphing them, namely, asymptotes, defined in layman's terms below.

When the graph of a function mimics a relation, we call the mimicry *asymptotic behavior*, and we call the mimicked relation an *asymptote*. If the asymptote is linear, we call it a *linear asymptote*.

For example, consider the reciprocal function  $y = 1/x$ . Since the reciprocal of large positive numbers greater than one are small proper fractions, the reciprocal function takes on small values as the  $x$ -values increase. Indeed, the larger the  $x$ -value, the smaller the  $y$ -value; hence, as the  $x$ -values approach infinity, the  $y$ -values approach zero. For large enough  $x$ -values, the function  $y = 1/x$  mimics the horizontal line  $y = 0$ . Hence,  $y = 0$  is an asymptote of the reciprocal function. In particular, we call  $y = 0$  a horizontal linear asymptote of  $y = 1/x$ . Similarly, the reciprocals of proper fractions are numbers greater than one. The closer a positive number is to

zero, the further its reciprocal is from zero; therefore, as  $x$ -values approach zero, the reciprocal function approaches infinity. Hence,  $x = 0$ , which describes a vertical line through the origin, serves as an asymptote of the reciprocal function. In particular, we call  $x = 0$  a vertical linear asymptote of  $y = 1/x$ .



The graphs of tangent, cotangent, secant, and cosecant all have domain restrictions (which is obvious from their definitions), and they all exhibit asymptotic behavior near these domain restrictions. For example, cosecant is undefined at  $x = \pi$  since it is the reciprocal of sine which equals zero at  $x = \pi$ . Since sine decreases to zero as  $x$  approaches  $\pi$  from the right, then its reciprocal, cosecant, must grow larger and larger as  $x$  approaches  $\pi$  from the right. Hence, the vertical line  $x = \pi$  serves as an asymptote for the graph of cosecant.

### Example Exercise

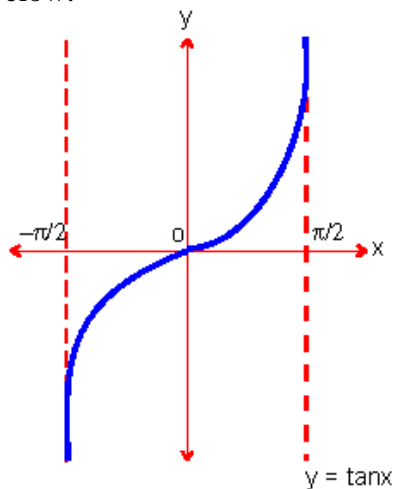
Graph  $y = \tan x$

We start by generating a table of sine and cosine values because we recall that tangent equals the ratio of sine to cosine (Fundamental Identity). From this table, we can find tangent's values.

$x$	$\cos x$	$\sin x$	$\frac{\sin x}{\cos x}$	$\tan x$
$-\pi/2$	0	-1	$-\frac{1}{0}$	undefined
$-\pi/3$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}/2}{1/2}$	$-\sqrt{3}$
$-\pi/4$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1/\sqrt{2}}{1/\sqrt{2}}$	-1
$-\pi/6$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1/2}{\sqrt{3}/2}$	$-\frac{1}{\sqrt{3}}$
0	1	0	$\frac{0}{1}$	0
$\pi/6$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1/2}{\sqrt{3}/2}$	$\frac{1}{\sqrt{3}}$
$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1/\sqrt{2}}{1/\sqrt{2}}$	1
$\pi/3$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}/2}{1/2}$	$\sqrt{3}$
$\pi/2$	0	1	$\frac{1}{0}$	undefined

If we extended our table, it would become obvious that tangent begins to repeat itself, so we see that tangent has a period equal to  $\pi$ . We are using  $[-\pi/2, \pi/2]$  as the fundamental cycle.

Plotting the ordered pairs from the table and suspecting vertical asymptotes where tangent is undefined, we generate the graph below.



### Suggested Homework

1. Graph each of the six trigonometric functions over their fundamental cycle. Compare the results with the graphs that appear in the *Function Gallery: Trigonometric Functions* on page xxii.

2. Graph each of the six trigonometric functions over the interval  $\left[-\frac{3}{2}\pi, \frac{3}{2}\pi\right]$ .

### Application Exercise

The frequency  $F$  of a sinusoidal wave with least period  $P$  is defined by  $F = \frac{1}{P}$ . The least period of  $y = \sin x$  is  $2\pi$ . The least period of  $y = \sin ax$  is  $\frac{2\pi}{|a|}$ .

What is the frequency of  $y = \sin ax$ ?