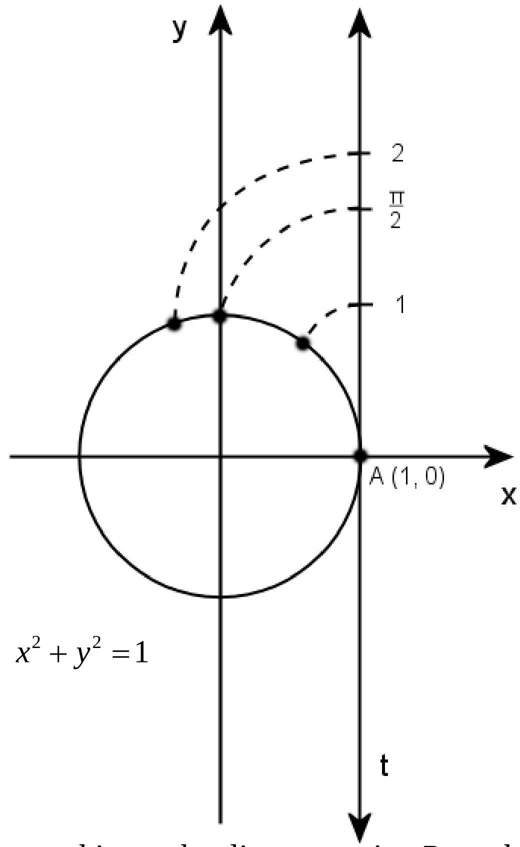
Unit Circle Winding Function

In this lecture, we want to construct a correspondence between the real number line and the unit circle, $x^2 + y^2 = 1$. We will visualize a vertical number line with its origin at the point $(1,0)$ on the unit circle.



Our goal is to map every number *t* on this number line to a point P on the unit circle in such a way that the arc length from $(1,0)$ to P on the circle represents the number's absolute value (allowing for multiple revolutions). This type of mapping would map the number π to the point $(-1,0)$ because the arc length along the unit circle from $(1,0)$ to $(-1,0)$ equals π. Similarly, we want to map 3π also to $(-1,0)$ since the arc length from $(1,0)$ to $(-1,0)$ equals 3π allowing for one complete revolution. Our mapping is actually wrapping or "winding" numbers from the number line around the unit circle. Indeed, we will call it a winding function and define it formally below.

Let *t* be any real number. Let W be a function that maps *t* to (x, y). We define W as

$$W(t) = \begin{cases} (1,0) & \text{if } t = 0 \\ (x, y) & \text{if } t \neq 0 \end{cases}$$

where (x, y) is on the circle $x^2 + y^2 = 1$ such that the arc length along the graph of $x^2 + y^2 = 1$ from $(1,0)$ to (x, y) equals $|t|$ with the revolution from $(1,0)$ to (x, y) being counterclockwise if $t > 0$ and clockwise if $t < 0$ (allowing for complete and multiple revolutions).

In other words, W maps any real number to a point on the unit circle so that Evaluating $W\left(-\frac{7\pi}{3}\right)$ requires finding the point P on the unit circle so that the arc length from $(1,0)$ to P equals $\left|-\frac{7\pi}{3}\right|$.

Obviously, P lies somewhere in the fourth quadrant. Dropping a perpendicular from P to the x-axis creates a right triangle with the radius of the circle serving as the hypotenuse. The reference angle for $-7\pi/3$ is an interior angle of the triangle. Since the reference angle for $-7\pi/3$ is $\pi/3$, we can employ the 30º-60º-90º Rule, to find the coordinates of point P, namely, $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$. Hence, $W\left(-\frac{7\pi}{3}\right)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

The winding function produces points on the unit circle, so it produces outputs of cosine and sine and leads to the following theorem.

Let t be any real number and let $W(t)=(x,y)$ be the winding function. Then, we have the following.

$$\cos t = x \text{ and } \sin t = y$$

In other words, $W(t)=(x,y)=(\cos t,\sin t)$.

This theorem is really an alternate definition of the cosine and sine functions, but it leads us directly to the *Pythagorean Identity*. Recall that if $W(t)=(x,y)$, then (x,y) is on the circle $x^2+y^2=1$. Since $W(t)=(x,y)=(\cos t,\sin t)$, we can substitute $\cos t$ and $\sin t$ for x and y respectively to obtain $(\cos t)^2+(\sin t)^2=1$. It is customary to write $(\sin t)^2$ as $\sin^2(t)$, so we have the Pythagorean Identity below.

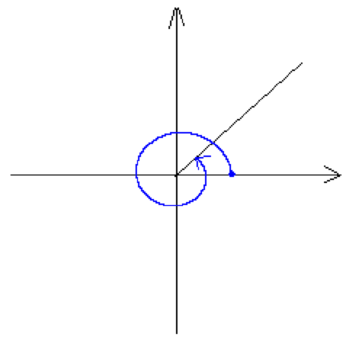
The *Pythagorean Identity* is $\cos^2(t)+\sin^2(t)=1$.

From the Pythagorean Identity, we see that $|\cos t|\le 1$ and $|\sin t|\le 1$. Hence, the range of both functions equals the interval $[-1,1]$.

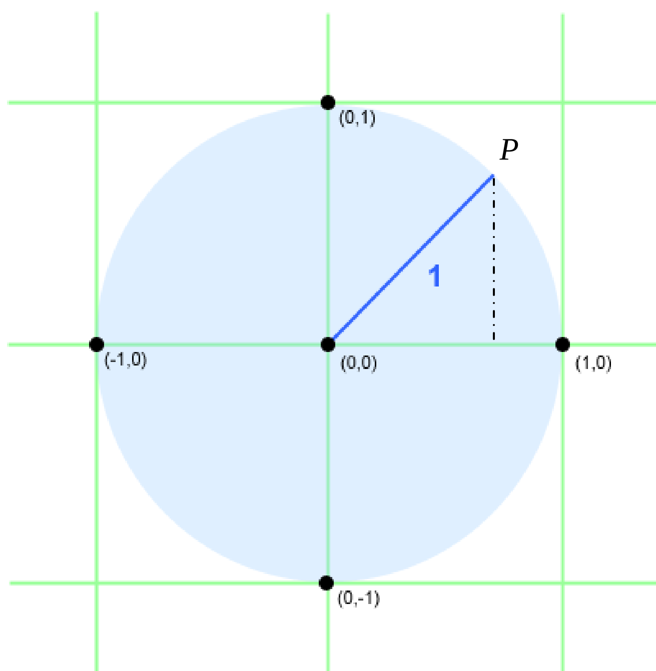
Example Exercise

Let $W(t)$ be the winding function defined in the lecture. Evaluate $W\left(\frac{9\pi}{4}\right)$.

Recall that 2π or $\frac{8\pi}{4}$ equals a full rotation of a circle. Obviously, an angle with a measure of $9\pi/4$ lies in the first quadrant, forming a reference angle of $\pi/4$.



We will call the intersection of the terminal side of this angle with the unit circle point *P*.



Dropping a perpendicular from point *P* to the *x*-axis creates a right triangle with the radius of the circle serving as the hypotenuse. The reference angle for $9\pi/4$ is an interior angle of the triangle. Since the reference angle for $9\pi/4$ is $\pi/4$ and since the degree measure of $\pi/4$ is $45°$, we can employ the Isosceles Right Triangle Rule, to find the coordinates of point *P*, namely, $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$. Hence, $W\left(\frac{9\pi}{4}\right)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

Suggested Homework

Let $W(t)$ be the winding function defined in the lecture. Evaluate $W\left(\frac{\pi}{6}\right)$, $W\left(\frac{\pi}{4}\right)$, $W\left(\frac{\pi}{3}\right)$, $W\left(\frac{\pi}{2}\right)$, $W\left(\frac{2\pi}{3}\right)$, $W\left(\frac{3\pi}{4}\right)$, $W\left(\frac{5\pi}{6}\right)$, $W(\pi)$, $W\left(\frac{7\pi}{6}\right)$, $W\left(\frac{5\pi}{4}\right)$, $W\left(\frac{4\pi}{3}\right)$, $W\left(\frac{3\pi}{2}\right)$, $W\left(\frac{5\pi}{3}\right)$, $W\left(\frac{7\pi}{4}\right)$, $W\left(\frac{11\pi}{6}\right)$, and $W(2\pi)$.

Application Exercise

A *rational point* on a plane curve is a point on the curve with rational coordinates. For example, $(-1,0)$ is a rational point on the circle with equation $x^2 + y^2 = 1$.

Let *t* be a natural number. Let $Q : t \rightarrow \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. Then, Q is a mapping from the natural numbers to the rational points on the circle $x^2 + y^2 = 1$, excluding the point $(-1,0)$. Use Q to find a rational point on the unit circle that is not on the *x*-axis or *y*-axis.