

## Combinations

Factorial provides a means for expressing the answer to any natural number multiplied by all the preceding natural numbers. The symbol for factorial is an exclamation point. Thus,  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ . Since zero is not a natural number,  $0!$  has no meaning according to the definition above, but we define  $0!$  as 1.

For any natural number  $n$ , we read the notation  $n!$  as “ $n$  factorial.” We define  $n!$  as below.

$$n! = n \cdot (n-1) \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

By convention, we say  $0! = 1$ .

Counting methods called permutations and combinations use factorials.

Combinations are selections of distinct objects in which the order of the objects is not important. If a club with twenty members, for example, is to select which three members must attend a national conference, the arrangements are combinations because choosing Alice, Bobby, and Charlene is not different from selecting Charlene, Bobby, Alice. In either case, the same three members will attend the conference.

The number of combinations of  $n$  objects taken  $r$  at a time is given by the *Combinations Theorem* below.

*Combinations Theorem:* If set  $S$  has  $n$  elements, then we denote the number of combinations of  $r$  distinct objects taken from  $S$  as  $C(n, r)$  or  ${}_n C_r$  or  $\binom{n}{r}$ . The formula below gives the number of combinations of  $r$  distinct objects taken from  $S$ .

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Consider a marketing director planning a promotion for a product sold in twelve different cities. If the director's budget only allows a promotional blitz in three cities, how many different ways can the director select three cities from the twelve target cities? The selection of Dallas, San Diego, and Nashville is no different from the selection of Nashville, San Diego, and Dallas because the order of the selection of those three cities does not change the fact that those three cities will be the focus of the marketing promotion. Since order does not make a difference, the *Combination Formula* gives the answer as follows.

$$C(12, 3) = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times \cancel{9!}}{3 \times 2 \times 1 \times \cancel{(9!)}} = \frac{\cancel{12}^2 \times 11 \times 10}{\cancel{3} \times 2} = 2 \times 11 \times 10 = 220.$$

## Suggested Homework

Section 11.1: #21-25 odd

Section 11.5: #5-13 odd

## Application Exercise

Given eight astronauts qualified as pilots and eighteen astronauts qualified as specialists, how many crews with one pilot and two specialists can be chosen for a mission?

## Binomial Expansion

When whole number exponents are applied to a binomial, the coefficients of the terms form a pattern.

$$\begin{aligned}
 (a + b)^0 &= && 1 \\
 (a + b)^1 &= && 1a + 1b \\
 (a + b)^2 &= && 1a^2 + 2ab + 1b^2 \\
 (a + b)^3 &= && 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\
 (a + b)^4 &= && 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \\
 (a + b)^5 &= && 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5
 \end{aligned}$$

The binomial expansion exhibits several patterns. First, the number of terms of each expansion equals  $n + 1$  where  $n$  equals the degree of the expansion (i.e., the exponent on the binomial). Second, the sum of the exponents in each term in the expansion equals the degree of the expansion. Third, the exponents on  $a$  in the expansion decrease by one, while the exponents on  $b$  increase by one. Fourth, the coefficients form a symmetrical pattern. Fifth, each entry below the second row is the sum of the closest pair of numbers in the line directly above it (other than the 1s at the end of each row).

These patterns give us the Binomial Theorem below.

If  $n$  is a natural number, then for any real numbers  $a$  and  $b$ , we have the following binomial expansion.

$$(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Recall that  $\binom{n}{k}$ , the number of combinations of  $n$  objects taken  $k$  at a time, equals  $\frac{n!}{k!(n-k)!}$ .

In general, the expression  $\binom{n}{k-1} a^{n-(k-1)} b^{k-1}$  equals the  $k$ th term of any binomial expansion.

## Suggested Homework

Section 11.5: #47, #49, #57

### Application Exercise

Let success be defined as any observable outcome of an experiment. Let  $\mathcal{W}$  represent exactly  $X$  number of occurrences of success in  $n$  trials. If the probability of success remains constant throughout  $n$  trials, then the formula below gives the probability of  $\mathcal{W}$ .

$$P(\mathcal{W}) = C(n, X) \cdot [p]^X \cdot [1-p]^{n-X}$$

where  $p$  equals the probability of success.

Only three out of four patients who undergo a particular surgical procedure are known to survive five years without having to undergo the surgery a second time. Of six patients who receive this treatment, what is the probability that all will survive five years without needing another surgery?