

Series

We use a special notation called *summation notation* as a shorthand way to indicate the sum of several terms in a sequence. The notation $\sum_{i=n}^m a_n$ indicates $a_n + a_{n+1} + a_{n+2} + \cdots + a_m$. The variable i , called the index of summation, assumes consecutive integer values beginning with n and ending with m . For example, consider $\sum_{k=0}^4 3^k$. In this example, the index of summation is k . The beginning value of the index is zero, and the terminating value is four. Each successive term is given by 3^k . Our example, $\sum_{k=0}^4 3^k$ reads, "The summation of 3^k as k progresses by consecutive integers from 0 to 4." To compute the sum, add each of the terms as shown here.

$$\begin{aligned}\sum_{k=0}^4 3^k &= 3^0 + 3^1 + 3^2 + 3^3 + 3^4 \\ \sum_{k=0}^4 3^k &= 1 + 3 + 9 + 27 + 81 \\ \sum_{k=0}^4 3^k &= 121\end{aligned}$$

Using summation notation, we define a very special type of sequence called a series defined in the box below.

Let $\{a_n\}_{n=k}^{\infty}$ be a sequence in \mathbb{R} . Construct the associated sequence $\{s_n\}_{n=k}^{\infty}$ as below.

$$\begin{aligned}s_k &= a_k \\ s_{k+1} &= a_k + a_{k+1} \\ s_{k+2} &= a_k + a_{k+1} + a_{k+2} \\ &\vdots \\ s_j &= a_k + a_{k+1} + \cdots + a_j \\ &\vdots \\ s_n &= a_k + a_{k+1} + \cdots + a_{j-1} + a_j + a_{j+1} + \cdots + a_n\end{aligned}$$

In other words, $s_n = \sum_{i=k}^n a_i$.

We call $\{s_n\}_{n=k}^{\infty}$ the *series* defined by the sequence $\{a_n\}$. Moreover, the number s_n is called the n th *partial sum* of the *infinite series* $\sum_{i=k}^{\infty} a_i$.

If a_n is arithmetic, we call s_n an arithmetic series. If a_n is geometric, we call s_n a geometric series.

Let's consider the series, $s_n = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$. This is a geometric series of the terms of the geometric sequence $\{a_n\}_{n=0}^{\infty} = \left\{\left(\frac{1}{2}\right)^n\right\}_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$. Hence, the terms of s_n are below.

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \dots$$

We obtain each term of the sequence by adding the appropriate number of terms of $\{a_n\}_{n=0}^{\infty}$. For a geometric series like this, we can use a formula to obtain any particular term of s_n .

Let s_n represent the sum of the first n terms of a geometric sequence $a_1 r^{n-1}$ as below.

$$s_n = a_1 r^{1-1} + a_1 r^{2-1} + a_1 r^{3-1} + \dots + a_1 r^{n-1}$$

If we multiply by the common ratio, we obtain the following.

$$\begin{aligned} r s_n &= a_1 r \cdot r^{1-1} + a_1 r \cdot r^{2-1} + a_1 r \cdot r^{3-1} + \dots + a_1 r \cdot r^{n-1} \\ r s_n &= a_1 r^1 + a_1 r^2 + a_1 r^3 + \dots + a_1 r^n \end{aligned}$$

Subtracting $r s_n$ from s_n eliminates most of the terms as below.

$$\begin{aligned} s_n - r s_n &= a_1 r^{1-1} + a_1 r^{2-1} + a_1 r^{3-1} + \dots + a_1 r^{n-1} - (a_1 r^1 + a_1 r^2 + a_1 r^3 + \dots + a_1 r^n) \\ s_n - r s_n &= a_1 r^0 + a_1 r^1 + a_1 r^2 + \dots + a_1 r^{n-1} - a_1 r^1 - a_1 r^2 - a_1 r^3 - \dots - a_1 r^n \\ s_n - r s_n &= a_1 r^0 - a_1 r^n \end{aligned}$$

Now, we factor and divide to obtain the sum of a finite geometric series.

$$\begin{aligned} s_n - r s_n &= a_1 r^0 - a_1 r^n \\ s_n (1 - r) &= a_1 (1 - r^n) \\ \frac{s_n (1 - r)}{(1 - r)} &= \frac{a_1 (1 - r^n)}{(1 - r)} \\ s_n &= \frac{a_1 (1 - r^n)}{(1 - r)} \end{aligned}$$

Thus, we have the theorem stated on the next page.

Sum of a Finite Geometric Series Theorem: If $s_n = \sum_{i=k}^n a_i$ and a_i is a geometric sequence, then

$$s_n = \frac{a_1(1-r^n)}{(1-r)}$$

For example, let's find the fifth partial sum of the series $s_n = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$. Since the index of summation starts at zero, to find the fifth partial sum, we let the index of summation progress to four. In other words, we need $\sum_{k=0}^4 \left(\frac{1}{2}\right)^k$. Plugging into the formula, we find s_5 as follows.

$$s_n = \frac{a_1(1-r^n)}{(1-r)}$$

$$s_5 = \frac{\left(\frac{1}{2}\right)^0 \left(1 - \left(\frac{1}{2}\right)^5\right)}{\left(1 - \frac{1}{2}\right)} = \frac{1 \left(1 - \frac{1}{32}\right)}{\frac{2}{2} - \frac{1}{2}} = \frac{\frac{32}{32} - \frac{1}{32}}{\frac{1}{2}} = \frac{\frac{31}{32}}{\frac{1}{2}} = \frac{31}{32} \div \frac{1}{2} = \frac{31}{32} \cdot 2 = \frac{31}{16}$$

Similarly, there exists a formula for a finite arithmetic series s_n as shown below.

$$s_n = a_1 + (a_1 + d) + (a_1 + 2d) + \dots + a_n$$

$$s_n = +a_n + (a_n - d) + (a_n - 2d) + \dots + a_1$$

$$2s_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n)$$

$$2s_n = n(a_1 + a_n)$$

$$s_n = \frac{n(a_1 + a_n)}{2}$$

Therefore, we have the sum of the first n terms of an arithmetic sequence.

Sum of a Finite Arithmetic Series Theorem: If $s_n = \sum_{i=k}^n a_i$ and a_i is an arithmetic sequence, then

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + d(n-1)]$$

For example, consider the sequence 2, 7, 12, 17, . . . To compute the sum of the first twenty terms, we can use $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}[2a_1 + d(n-1)]$. We will use the latter formula since we do not have a_{20} (although we could find it easily).

$$S_n = \frac{n}{2}[2a_1 + d(n-1)]$$
$$S_{20} = \frac{20}{2}[2(2) + 5(20-1)] = 10[4 + 5(19)] = 990$$

Both of the previous two theorems provide formulas for the value of a finite series. It seems counterintuitive to think we could find the value of an infinite series, but sometimes this is the case. For our purposes, we will simply accept the following theorem without proof.

Sum of an Infinite Geometric Series Theorem: If s_∞ is an infinite geometric series, $a + ar + ar^2 + ar^3 + \dots$, with $|r| < 1$, then

$$s_\infty = \frac{a}{1-r}.$$

For example, consider the series $300 + 225 + \frac{675}{4} + \dots$. We calculate the sum below.

$$S = \frac{a_1}{1-r} = \frac{300}{1-.75} = \frac{300}{.25} = 1,200$$

Suggested Homework

Section 11.2: #1-25 odd, #53-65 odd

Section 11.3: #61, #63, #77, #81

Application Exercise

If a patient starts taking 100 mg of a particular drug every 8 hours, then $\sum_{k=1}^n 100(0.69)^{k-1}$ gives the amount of the drug in the patient's bloodstream after taking the n th pill. Calculate the amount of the drug in the patient's bloodstream after 25 pills.