Sequences

In general, a sequence is a list of numbers indexed by the natural numbers. The list, 1, 3, 5, 7, 9, . . . is an example of an arithmetic sequence. The ellipsis (the three dots) indicates that the pattern will continue according to the establish pattern signified in the first few elements of the sequence. In the example 1, 3, 5, 7, 9, . . . the sequence grows by a common difference of two, so the next number after nine will be eleven. Each number in the sequence is called a term. In the sequence 1, 3, 5, 7, 9, . . . the first element, 1, is the first term. The second element, 3, is the second term while 5 is the third term, and so on. The notation a_1, a_2, a_3, . . ., denotes the different terms in a sequence. Thus, a_{15} denotes the fifteenth term in a sequence. The expression a_n is referred to as the general or *n*th term of the sequence.

Considering the notation just discussed, the formal definition of a sequence given below makes sense.

> A sequence, denoted $\{a_n\}_{n=m}^{\infty}$, is a function that has a specified value for each integer $n \geq m$ where m usually equals 1 or 0.

Above we define a sequence to be a function that maps the natural numbers (or sometimes the nonnegative integers) to real numbers.

An arithmetic sequence is a progression of numbers that grow with regular spacing, that is, they grow with a common difference. For example, 6, 9, 12, 15, . . . is an arithmetic sequence with a common difference of three. Each number is three higher than the number that precedes it.

> Let $\{a_n\}$ be a sequence of real numbers. If $\{a_n\} = \{a_1 + d \cdot (n-1)\}$, then $\{a_n\}$ is an *arithmetic sequence* with common difference equal to *d*.

A geometric sequence is a progression of numbers that grow with regular *relative* spacing, that is, they grow (or diminish) according to multiplication by a common ratio. For example, 7, 14, 28, 56, . . . is a geometric sequence with a common ratio of two. Each number is double the previous number. Geometric sequences grow/diminish much more rapidly than arithmetic sequences.

> Let $\{a_n\}$ be a sequence of real numbers. Let $r > 0 \wedge r \neq 1$. If $\{a_n\} = \{a_1 \cdot r^{n-1}\}$, then $\{a_n\}$ is a *geometric sequence* with *common ratio* equal to r.

A sequence progresses (or regresses) according to a rule, but the rule operates on a domain restricted to the natural numbers (sometimes the whole numbers). For example, a sequence could progress according to the rule $\{a_n\}_{n=1}^{\infty} = \{5n+2\}_{n=1}^{\infty}$ where *n* represents the natural numbers and a_n represents the *n*th term of the sequence. We An astute student might note the similarity with a linear function. This similarity exists because the sequence exemplifies an arithmetic sequence, which progresses according to a common difference. The slope of a line serves as a common difference creating the "regular spacing" (the common difference) between one term and the next. Examine the first three terms of the sequence $a_n = 5n + 2$.

$$a_1 = 5(1) + 2 = 7$$
$$a_2 = 5(2) + 2 = 12$$
$$a_3 = 5(3) + 2 = 17$$

The first three terms of this sequence are 7, 12, 17. Adding five to a term predicts the subsequent term. Thus, the fourth term of the sequence would be 22 because 17 + 5 = 22. Sequences, like this one, that progress according to a common difference are arithmetic.

The rule of a sequence often involves a common difference as shown in the above example or a common ratio. Sequences that progress according to a common ratio are geometric sequences. For example a geometric sequence might progress according to the rule $a_n = 4(3^n)$ where n represents the natural numbers and a_n represents the nth term of the sequence. An astute student might note the similarity with an exponential function. This similarity exists because the sequence progresses according to a common ratio, that is, a constant factor that is multiplied by the previous term to produce the next term, creating a "regular *relative* spacing." Examine the first three terms of the sequence $a_n = 4(3)^n$.

$$a_1 = 4(3)^1 = 12$$
$$a_2 = 4(3)^2 = 36$$
$$a_3 = 4(3)^3 = 108$$

The first three terms of this sequence are 12, 36, 108. Multiplying a term by three predicts the subsequent term. Thus, the fourth term of the sequence would be 324 because 3 × 108 = 324. The factor, in this case 3, is a "common ratio" because it equals the ratio of a term and the previous term. Sequences, like this one, that progress according to multiplication are geometric.

The General Term of Sequences

An arithmetic sequence is a sequence in which adding the same value to the previous term finds each successive term (after the second term). Its general term is described by the formula $a_n = a_1 + (n - 1)d$. The number d is the common difference. Taking any term in the sequence and subtracting its preceding term finds the common difference, d. Consider the sequence, 6, 13, 20, 27, . . . Find the common difference by taking any term after the first and subtracting the preceding term. Thus, the common difference is seven because 20 – 13 = 7. The equation $a_n = a_1 + (n - 1)d$ gives any particular term of an arithmetic sequence, so the general term of the sequence 6, 13, 20, 27, . . . is as follows.

$$
\begin{aligned}
a_n &= a_1 + (n-1)d \\
a_n &= 6 + (n-1)7 \\
a_n &= 6 + 7n - 7 \\
a_n &= 7n - 1
\end{aligned}
$$

We find the twentieth term in particular below.

$$
\begin{aligned}
a_n &= 7n - 1 \\
a_{20} &= 7(20) - 1 \\
a_{20} &= 140 - 1 \\
a_{20} &= 139
\end{aligned}
$$

A geometric sequence is a sequence in which multiplying a term by the same factor finds the next term. Its general term is $a_n = a_1 r^{n-1}$. The value r is the common ratio. Taking any term in the sequence and dividing by its preceding term gives r. Consider the sequence 40, 80, 160, 320, . . . Find the common ratio by finding the ratio of a term to its preceding term. Thus, the common ratio is two because $r = \frac{a_2}{a_1} = \frac{80}{40} = 2$. The equation $a_n = a_1 r^{n-1}$ gives any particular term of a geometric sequence, so the general term of the sequence 40, 80, 160, 320, . . . is below.

$$
\begin{aligned}
a_n &= a_1 r^{n-1} \\
a_n &= 40(2)^{n-1}
\end{aligned}
$$

In particular, we find the seventh term as follows.

$$
\begin{aligned}
a_7 &= 40(2)^{7-1} \\
a_7 &= 40 \cdot 2^6 \\
a_7 &= 40 \cdot 64 \\
a_7 &= 2{,}560
\end{aligned}
$$

Recursive sequences are sequences in which later terms depend upon earlier terms. Defining the first term or the first few terms then writing an equation that depends on these definitions creates a recursive formula for the *n*th term of a recursive sequence. Consider the sequence 8, 18, 33, 53, . . . Here, the first term is defined as $a_1 = 8$. Then, the equation $a_n = a_{n-1} + 5n$ defines the sequence. The fifth term is below.

$$
\begin{aligned}
a_n &= a_{n-1} + 5n \\
a_5 &= a_4 + 5(5) \\
a_5 &= 53 + 25 \\
a_5 &= 78
\end{aligned}
$$

Suggested Homework

Section 11.1: #13, #15, #63-73 odd
Section 11.3: #1-13 odd, #21-29 odd

Application Exercise

A retirement plan gives the recipient a fixed raise of d dollars each year. If the fifth year income was $\$24,500$ and the ninth year income was $\$25,700$, then what was the first year income?