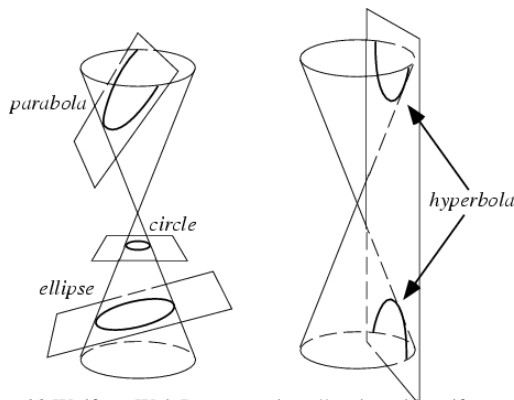


## The Conic Sections

The conic sections include a family of curves generated when planes intersect a cone. These curves include the circle, ellipse, parabola, and hyperbola. The online encyclopedia *MathWorld* illustrates the conic sections with the graphic below.



Weisstein, Eric W. "Conic Section." *MathWorld*. Wolfram Web Resource. <http://mathworld.wolfram.com/ConicSection.html>

## Parabolas

The intersection of a cone and a plane parallel to one of the straight lines that generate the cone is a parabola. More precisely, a parabola is the set of points in a plane such that the distance of each point in the set from a fixed point called the focus equals the perpendicular distance from the point to a fixed line called the directrix. Since the set of points *equidistant* from a fixed point and a fixed line defines a parabola, the *distance* formula will give the standard equation of a parabola. The focus lies on the parabola's axis of symmetry, and the directrix is perpendicular to the parabola's axis of symmetry. The vertex is the point of the parabola that lies on the axis of symmetry. The vertex is an equal distance from the focus and the nearest point on the directrix according to the definition of a parabola. Let the vertex be the point  $(h,k)$ . Assume the parabola has a vertical axis of symmetry. Let  $c$  be the distance from the vertex to the focus. The focus has the coordinates  $(h, k + c)$ . The equation  $y = k - c$  describes the directrix. To get the equation of the set of points that are the same distance from the line  $y = k - c$  and the point  $(h, k + c)$ , choose any point  $P$  with coordinates  $(x, y)$  such that the distance from  $P$  to the focus equals the length of a perpendicular line segment stretching from  $P$  to  $y = k - c$ . This line segment intersects  $y = k - c$  at a point  $D$  with the coordinates  $(x, k - c)$ . Since the distance from  $P$  to the focus must equal the distance from  $P$  to  $D$ , the distance formula gives an equation of a parabola.

$$\sqrt{(x-h)^2 + (y-(k+c))^2} = \sqrt{(x-x)^2 + (y-(k-c))^2}$$

Simplify and square both sides of the equation.

$$\begin{aligned}\sqrt{(x-h)^2 + (y-(k+c))^2} &= \sqrt{(y-(k-c))^2} \\ \left[ \sqrt{(x-h)^2 + (y-(k+c))^2} \right]^2 &= \left[ \sqrt{(y-(k-c))^2} \right]^2 \\ (x-h)^2 + (y-(k+c))^2 &= (y-(k-c))^2\end{aligned}$$

Solve for  $y$ .

$$\begin{aligned}(x-h)^2 + (y-(k+c))^2 &= (y-(k-c))^2 \\ (x-h)^2 + y^2 - 2y(k+c) + (k+c)^2 &= y^2 - 2y(k-c) + (k-c)^2 \\ (x-h)^2 - 2yk - 2yc + k^2 + 2kc + c^2 &= -2yk + 2yc + k^2 - 2kc + c^2 \\ (x-h)^2 + 4kc &= 4yc \\ y &= \frac{(x-h)^2}{4c} + k \\ y &= \frac{1}{4c} \cdot (x-h)^2 + k\end{aligned}$$

Let  $\frac{1}{4c} = a$  to obtain the standard form of a parabola with a vertical axis of symmetry.

$$\begin{aligned}y &= \frac{1}{4c} \cdot (x-h)^2 + k \\ y &= a(x-h)^2 + k\end{aligned}$$

Accordingly, the equation in standard form of a parabola with vertex  $(h, k)$  and the vertical line  $x = h$  as the axis of symmetry appears in the box below.

Let  $(x, y)$  be a point on a parabola with vertex  $(h, k)$  and the vertical line  $x = h$  as the axis of symmetry, then

$$y = a(x-h)^2 + k$$

where  $(h, k)$  is the vertex and where  $a \cdot c = 1/4$  and  $c$  represents the distance from the vertex to the focus.

The equation in standard form of a parabola with vertex  $(h, k)$  and the horizontal line  $y = k$  as the axis of symmetry appears in the box on the following page.

Let  $(x, y)$  be a point on a parabola with vertex  $(h, k)$  and the horizontal line  $y = k$  as the axis of symmetry, then

$$x = a(y - k)^2 + h$$

where  $(h, k)$  is the vertex and where  $a \cdot c = 1/4$  and  $c$  represents the distance from the vertex to the focus.

To graph a parabola, it is convenient to know its vertex. Consider the equation in quadratic form,  $y = -2x^2 - 4x + 6$ . The graph of a quadratic (or second-degree polynomial) is a parabola. The parabola has a  $y$ -intercept at  $(0, 6)$ . The  $x$ -intercepts occur where  $y = 0$ , so solving the equation  $-2x^2 - 4x + 6 = 0$  gives the  $x$ -intercepts.

$$-2x^2 - 4x + 6 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x - 1)(x + 3) = 0$$

$$x - 1 = 0, x + 3 = 0$$

$$x = 1, x = -3$$

$$(1, 0) \text{ and } (-3, 0)$$

The vertex, however, is not self-evident since the equation appears in quadratic form. Completing the square can transform the parabola, however, to standard form. The goal is to obtain an equivalent equation of the form  $y = a(x - h)^2 + k$  for  $y = -2x^2 - 4x + 6$ . Write the equation.

$$y = -2x^2 - 4x + 6$$

Ignore the constant and factor out the leading term from the binomial.

$$y = -2x^2 - 4x + 6$$

$$y = -2(x^2 + 2x) + 6$$

Use the binomial within the parenthesis to compose a perfect square trinomial in  $x$ . Recall that  $(a \pm b)^2 = a^2 \pm 2ab + b^2$ . In a perfect square trinomial, the linear term equals twice the product of the two terms of the binomial. To complete a square, therefore, the constant term must be the square of half the linear term. To complete the square, add the square of half the linear term,  $[\frac{1}{2}(2)]^2$ , which equals 1. Adding one within the parenthesis actually adds negative two to the right side of the equation, so add negative two to the left side of the equation.

$$y = -2(x^2 + 2x) + 6$$

$$y - 2 = 2(x^2 + 2x + 1) + 6$$

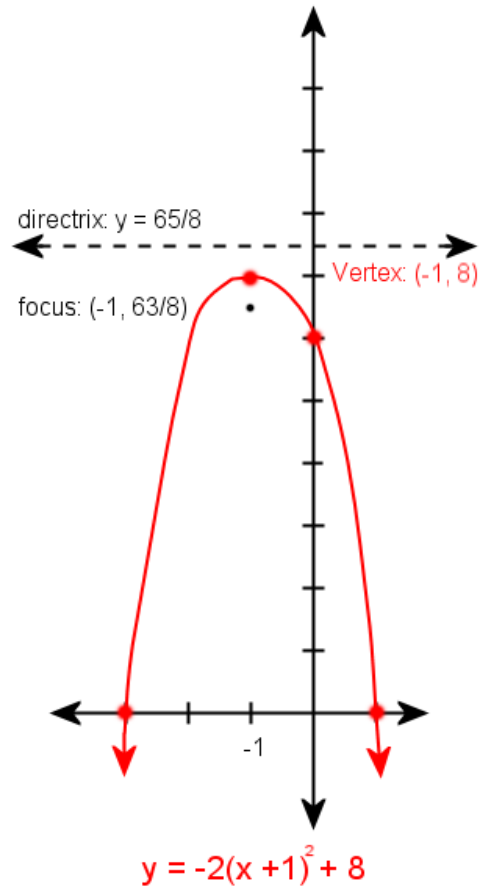
Factor the perfect square trinomial to a squared binomial.

$$y - 2 = -2(x + 1)(x + 1) + 6$$

$$y - 2 = -2(x + 1)^2 + 6$$

$$y = -2(x + 1)^2 + 8$$

Now the equation is in standard form, and the vertex can be identified as  $(-1, 8)$ .



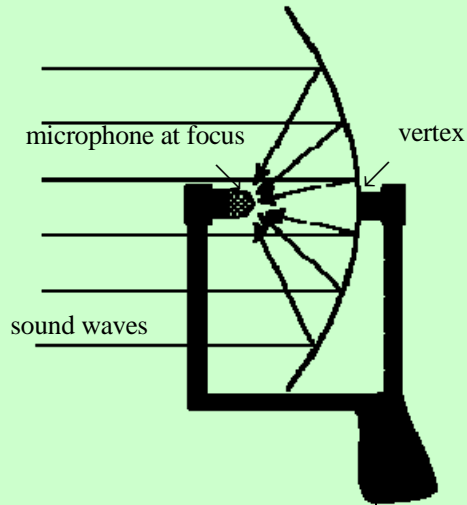
The focus has the coordinates  $(h, k + c)$ , and the directrix has the equation  $y = k - c$ . Since  $a \cdot c = 1/4$ ,  $c = -1/8$ . Thus, the focus is the point  $(-1, 63/8)$  and the equation  $y = 65/8$  describes the directrix.

## Suggested Homework

Section 10.1: #19, #27-35 odd, #51-59 odd

### Application Exercise

Eavesdroppers use a microphone placed at the focus of a parabolic shield as shown below in order to “pull in voices” from far away.



Let the equation  $x = -\frac{1}{12}y^2$  trace out the parabola of the shield. How far from the vertex of the shield should the microphone appear to maximize it's effectiveness?

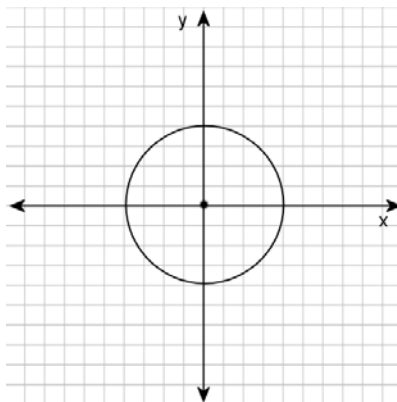
## Circles

The intersection of a cone and a plane perpendicular to the axis of the cone is a circle. A circle is a set of points in a plane, each of which is equidistant from a fixed point called the center. The distance between the points of the circle and the center is the radius. Since the set of points *equidistant* from a fixed point defines a circle, the *distance* formula will give the standard equation of a circle. Suppose that a circle has center  $(h,k)$  and radius  $r > 0$ . If  $(x,y)$  is any point on the circle, then we can use the distance formula to write  $\sqrt{(x-h)^2 + (y-k)^2} = r$ . Squaring both sides of this equation gives the standard equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

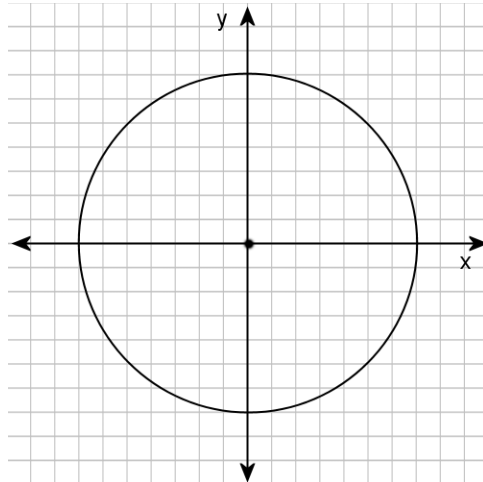
This equation can be expanded as shown here:  $x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = r^2$ . This expansion is called the general form of the equation of a circle when  $-2h$  equals  $c$ ,  $-2k$  equals  $d$ , and  $k^2 - r^2$  equals  $e$ , so that the general form is  $x^2 + y^2 + cx + dy + e = 0$  where  $c$ ,  $d$ , and  $e$  are real numbers.

To graph a circle, affix an endpoint of a line segment whose length equals the radius at the center then rotate the segment about  $360^\circ$ . Consider a circle described by the equation  $x^2 + y^2 = 16$ . The circle has a radius of 4 and a center at  $(0,0)$ . The graph appears below.

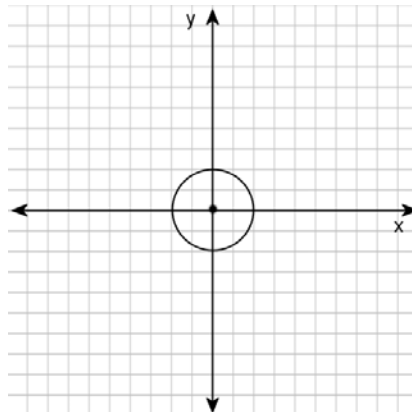


The points that compose the circle satisfy the equation  $x^2 + y^2 = 16$ , and each point is 4 units from the center  $(0,0)$ .

If the radius increases, the circle widens. Consider the circle described by  $x^2 + y^2 = 49$  with a radius of 7 and a center at  $(0,0)$ . The graph is below.

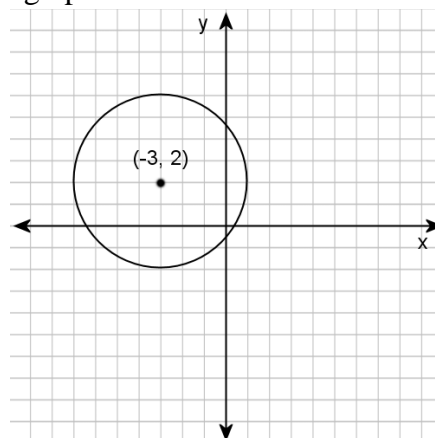


Similarly, if the radius decreases, the circle tightens. Consider the circle described by  $x^2 + y^2 = 4$  with a radius of 2 and a center at  $(0,0)$ . The graph is below.



If the center moves, the entire circle moves, creating a translation of the circle with center at the origin.

Recall the standard equation of a circle,  $(x - h)^2 + (y - k)^2 = r^2$  where  $(h,k)$  is the center and  $r$  is the radius. Consider the original example,  $x^2 + y^2 = 16$ , which has its center at  $(0,0)$ . If the center moves from  $(0,0)$  to  $(-3,2)$ , the equation changes from  $x^2 + y^2 = 16$  to  $(x + 3)^2 + (y - 2)^2 = 16$ , and the graph moves as shown below.



To graph a circle, it is convenient to know its center and radius. Consider the equation in general form,  $x^2 + y^2 - 6x + 4y - 3 = 0$ . The center is not self-evident when the equation is written in general form. This equation can be written in the standard form following a process called completing the square. The goal is to obtain an equivalent equation of the form  $(x - h)^2 + (y - k)^2 = r^2$  for  $x^2 + y^2 - 6x + 4y - 3 = 0$ . Write the equation with the constant on the right.

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

$$x^2 + y^2 - 6x + 4y = 3$$

Use the commutative property to gather like variables.

$$x^2 + y^2 - 6x + 4y = 3$$

$$x^2 - 6x + y^2 + 4y = 3$$

Compose a perfect square trinomial in both  $x$  and  $y$ . Creating perfect square trinomials is called completing the square. Recall that  $(a \pm b)^2 = a^2 \pm 2ab + b^2$ . The middle term is frequently a linear term, and the last term is frequently a constant. In a perfect square trinomial, the linear term equals twice the product of the two terms of the binomial. To complete a square, therefore, the constant term must be the square of half the linear term. To complete the square in  $x$ , add the square of half the linear term,  $[\frac{1}{2}(-6)]^2$ . To complete the square in  $y$ , add the square of half the linear term,  $[\frac{1}{2}(4)]^2$ .

$$x^2 - 6x + y^2 + 4y = 3$$

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 3 + 9 + 4$$

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 16$$

Factor the trinomials to squared binomials.

$$x^2 - 6x + 9 + y^2 + 4y + 4 = 16$$

$$(x - 3)(x - 3) + (y + 2)(y + 2) = 16$$

$$(x - 3)^2 + (y + 2)^2 = 16$$

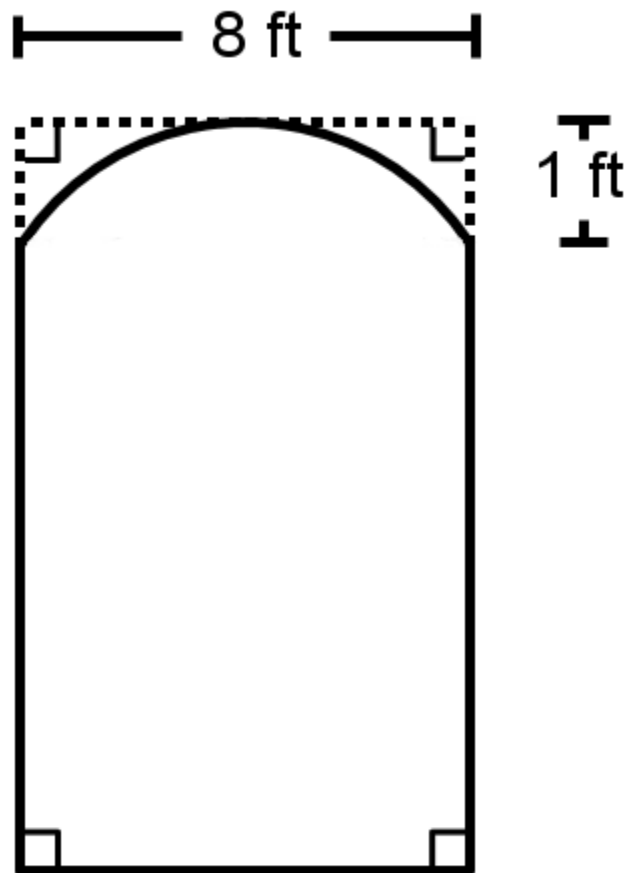
Now the equation is in standard form, and the center can be identified as  $(3, -2)$  and the radius as four.

Suggested Homework

Section 10.2: #47-63 odd

Application Exercise

Find the radius of the circle that contains the arch shown in the diagram below.  
HINT: Impose a coordinate system.



## Ellipses

The box below contains the equation of an ellipse.

An ellipse centered at  $(h, k)$  with horizontal major axis of length  $2a$  and vertical minor axis of length  $2b$  has the equation below.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Similarly, an ellipse centered at  $(h, k)$  with vertical major axis of length  $2a$  and horizontal minor axis of length  $2b$  has the equation below.

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

An ellipse has a pair of foci that lie on the major axis. The foci of ellipses described by the equations in the above box lie  $\pm c$  units from the center  $(h, k)$ . The equation  $c^2 = a^2 - b^2$  yields  $c$ .

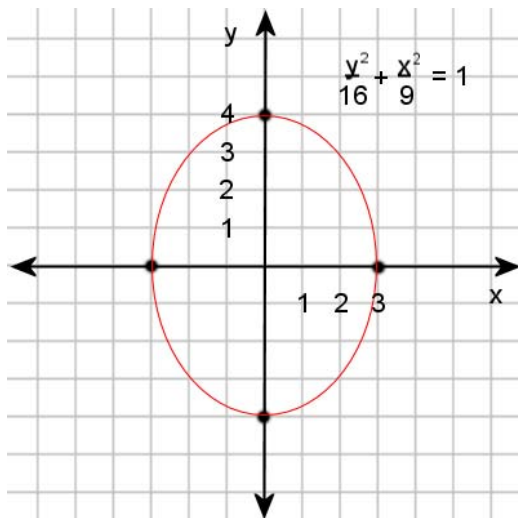
To graph an ellipse it is convenient to know the length of the major and minor axes and to know its center. It is also important to find the foci since the definition of the ellipse depends upon the focal points. Consider the ellipse described by the equation  $y^2/16 + x^2/9 = 1$ . The equation shows that  $h$  and  $k$  equal zero, so the center occurs at the origin. The equation also shows that  $a^2 = 16$  and  $b^2 = 9$ ; therefore,  $a = 4$  and  $b = 3$ . The major axis is vertical and it extends from  $(0, -4)$  to  $(0, 4)$ . The minor axis is horizontal and it extends from  $(-3, 0)$  to  $(3, 0)$ . The equation  $c^2 = a^2 - b^2$  obtains the distance from the center to the foci as shown here.

$$c^2 = a^2 - b^2$$

$$c^2 = 16 - 9$$

$$c = \pm\sqrt{7}$$

Consequently, the foci occur at  $(0, \sqrt{7})$  and  $(0, -\sqrt{7})$ . The graph of the ellipse is below.



Let's assume we need to identify the foci of an ellipse with the equation below.

$$\frac{(x+2)^2}{16} + \frac{(y-2)^2}{25} = 1$$

First, we rewrite the equation in standard form.

$$\frac{(x-(-2))^2}{4^2} + \frac{(y-2)^2}{5^2} = 1$$

Since  $a > b$  and  $5 > 4$ , a cursory examination of the given equation reveals that it describes an ellipse with a vertical major axis. The center of the ellipse is  $(-2, 2)$  since  $h = -2$  and  $k = 2$ . The major axis lies on the line  $x = -2$ .

The foci of an ellipse are always  $c$  units from the center on the major axis where  $c^2 = a^2 - b^2$  and  $a > b$ . Substituting for  $a$  and  $b$ , we obtain the following.

$$c^2 = 25 - 16$$

$$c^2 = 9$$

$$c = \pm 3$$

Since the major axis is vertical, add  $\pm 3$  to  $k$ , the  $y$ -value of the center, to find the foci. The foci are  $(-2, -1)$  and  $(-2, 5)$ .

The box below defines the eccentricity  $e$  of an ellipse.

Let  $e$  represent the eccentricity of an ellipse whose foci are  $c$  units from the center and whose major axis equals  $2a$  units in length. Then, we define eccentricity as below.

$$e = c/a.$$

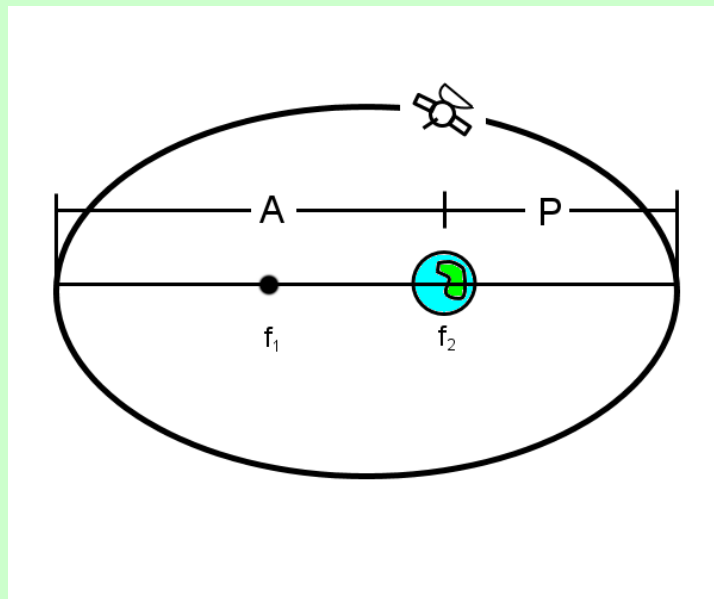
Since  $0 < c < a$ , we have  $0 < e < 1$ . For an ellipse that appears circular, the foci are close to the center. Hence  $c$  is small compared to  $a$ , so the eccentricity of a nearly circular ellipse is near zero. Elongated ellipses have eccentricity near one.

## Suggested Homework

Section 10.2: #25-33 odd, #37

### Application Exercise

For a satellite in an elliptical orbit around a planet, perigee,  $P$ , is defined to be its closest distance to the planet and apogee,  $A$ , is defined to be its greatest distance from the planet.



Let  $e$  represent the eccentricity of an ellipse. Show that  $e = \frac{A - P}{A + P}$ .