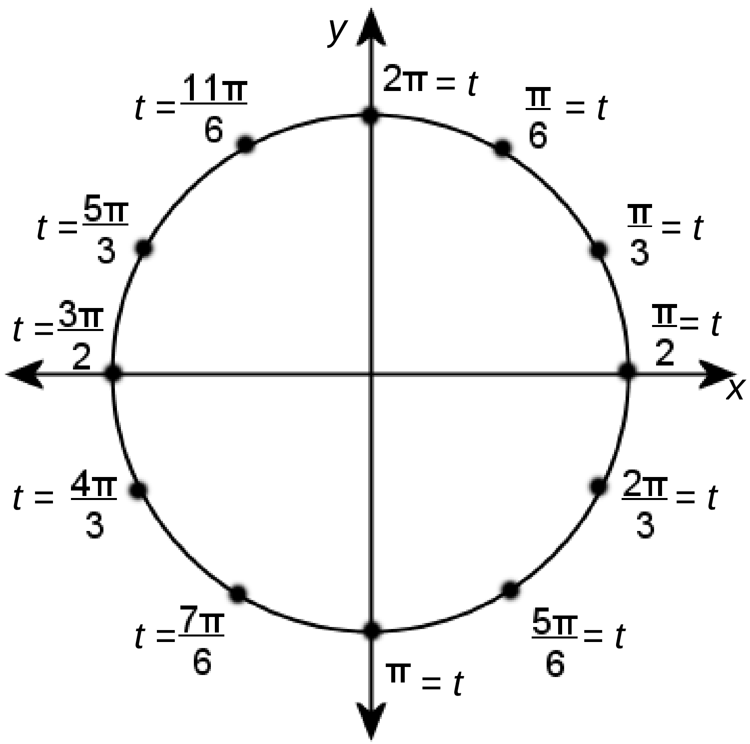
Parametric Equations

We are familiar with functions of the form $y = f(x)$ or $z = f(x, y)$. In either case, we have a direct relationship between *x* and *y*. Sometimes, however, expressing *x* and *y* in terms of a third variable called a *parameter* is convenient. In other words, using t as the parameter, we let $x = f(t)$ and $y = g(t)$, and we call these equations *parametric equations*.

For instance, consider the parametric equations $x = \sin t$ and $y = \cos t$. If $t = 0$, then $x = 0$ and $y = 1$. Thus, the point $(0,1)$ is a point of the graph described parametrically by $x = \sin t$ and $y = \cos t$. Note that we still use the *x*-axis and *y*-axis in the coordinate system. The parameter *t* does not appear on the graph. Finding more points, we generate the table and graph below.

t	$x = \sin t$	$y = \cos t$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$
π	0	-1
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$
$3\pi/2$	-1	0
$5\pi/3$	$-\sqrt{3}/2$	$1/2$
$11\pi/6$	$-1/2$	$\sqrt{3}/2$
2π	0	1



It looks as if we are generating the unit circle. To prove this, we can recall our unit circle definitions for sine and cosine, or we can eliminate the parameter. There are many tricks to eliminating a parameter, one is to square both equations as below.

$$x = \sin t \qquad y = \cos t$$
$$x^2 = \sin^2 t \qquad y^2 = \cos^2 t$$

Adding the two equations and applying the Pythagorean Identity, we conclude that the function is indeed a circle with radius equal to one.

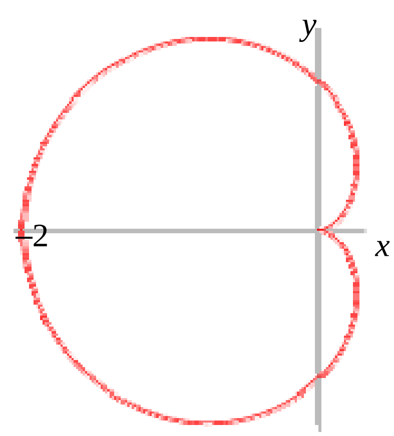
$$x^2 + y^2 = \sin^2 t + \cos^2 t$$
$$x^2 + y^2 = 1$$

Consider the parametric equations $x = t + 1$ and $y = t^2 + 3t + 2$. We could sketch the graph point-by-point using a table of incremental *t*-values and the corresponding *x* and *y*-values, or we can eliminate the parameter to return to the more familiar rectangular representation. Solving the first equation for *t* gives us $t = x - 1$. Then, we conclude by substituting $x - 1$ for t into $y = t^2 + 3t + 2$ as below.

$$y = t^2 + 3t + 2$$
$$y = (x-1)^2 + 3(x-1) + 2$$
$$y = x^2 - 2x + 1 + 3x - 3 + 2$$
$$y = x^2 - 2x + 3x + 1 - 3 + 2$$
$$y = x^2 + x$$
$$y = x(x+1)$$

Hence, we see that we have a familiar second-degree polynomial with *x*-intercepts at $x = -1$ and $x = 0$.

Parametric equations provide a means for representing polar graphs in the rectangular coordinate system. Consider the cardoid $r = 1 - \cos\theta$ and recall the equations $x = r\cos\theta$ and $y = r\sin\theta$. Substituting $1 - \cos\theta$ for *r* into these two equations, we obtain $x = (1 - \cos\theta)\cos\theta$ and $x = (1 - \cos\theta)\sin\theta$, which gives a pair of parametric equations using θ as the parameter. These equations describe a cardoid in the rectangular coordinate system.

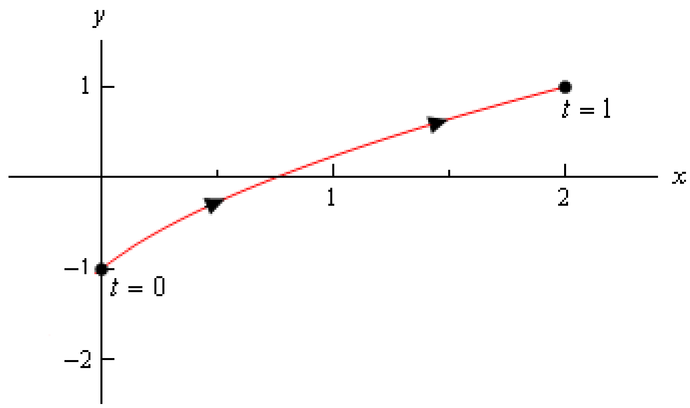


A natural question at this point might ask what advantage we gain by defining a relation with parametric equations. Well, one advantage is the direction of motion suggested by the parameter. For instance, consider the parametric equations below, which describe the motion of a particle in a plane.

$$x = t^2 + t \text{ and } y = 2t - 1$$

If we assume t represents seconds elapsed with $t = 0$ corresponding to the particle's original position, then when we plot points on the curve using incremental t-values, the resulting curve traces out the motion of the particle as time elapses over a one-second interval as shown below.

t	x	y
0	0	–1
0.5	0.75	0
1	2	1



Suggested Homework

Section 7.7: #7, #13-19 odd

Application Exercise

Suppose that the parametric equations below describe the position of one particle at time t .

$$x_1 = 3\sin t,\ y_1 = 2\cos t,\ 0 \le t \le 2\pi$$

Suppose further that the next pair of parametric equations describe the position of a second particle at time t.

$$x_2 = -3 + \cos t,\ y = 1 + \sin t,\ 0 \le t \le 2\pi$$

Graphs the paths of both particles and determine the number of literal intersection points in the paths of the particles. Will the particles collide?