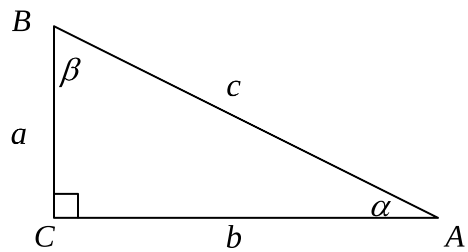
Trigonometric Functions Defined

From algebra, we are familiar with the idea of a function, a set of ordered pairs (x, y) generated by a rule or *mapping* that pairs every distinct *x*-value with one and only one *y*-value. We call the *x*-values *inputs* or domain values and the *y*-values *outputs* or range values. For instance, if $f(2)=8$, then f is a rule such that the input 2 is mapped to the output 8, creating the ordered pair (2,8).

In trigonometry, we study functions whose inputs are often thought of as the measures of interior angles of a right triangle and whose outputs are the ratios of side lengths of the triangle. To generate an example, let's consider the right triangle below.



In the diagram of $\triangle ABC$, the length of the hypotenuse equals *c*. $\angle A$ measures α while $\angle B$ measures β. The length of leg opposite $\angle A$ equals *a* while the length of the leg opposite $\angle B$ measures *b*. Using the measure of $\angle A$ as an input, we have six different ratios to discuss as listed below.

$$\frac{a}{c}, \frac{c}{a}, \frac{b}{c}, \frac{c}{b}, \frac{a}{b}, \wedge \frac{b}{a}$$

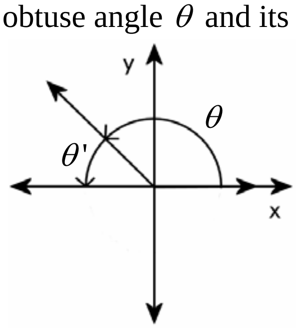
To discuss these ratios, we will define six trigonometric functions listed below respectively.

$$sine, cosecant, cosine, secant, tangent, \wedge\ cotangent$$

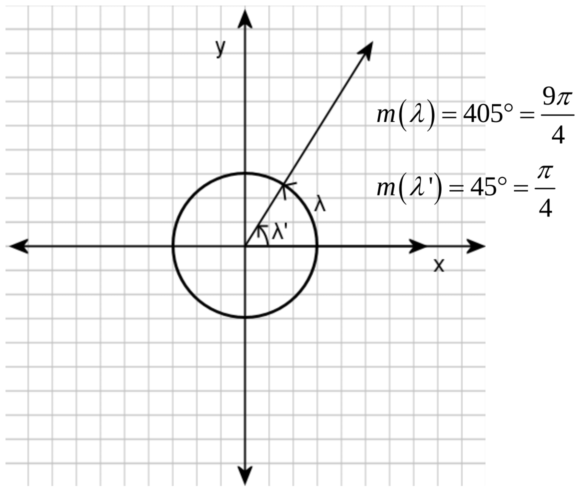
We said that the inputs of these functions are *usually* thought of as interior angles of a right triangle. This way of thinking turns out to be too restrictive since the interior angles of a right angle are restricted to measures on the interval $[0, \pi/2]$ (where zero represents the degenerate case). To eliminate this restriction, we first define *reference angles* as below.

If θ is a nonquadrantal angle in standard position, then the reference angle for θ is the positive acute angle θ' formed by the terminal side of θ and the *x*-axis.

In the diagram below, we have the obtuse angle θ and its acute reference angle θ'.



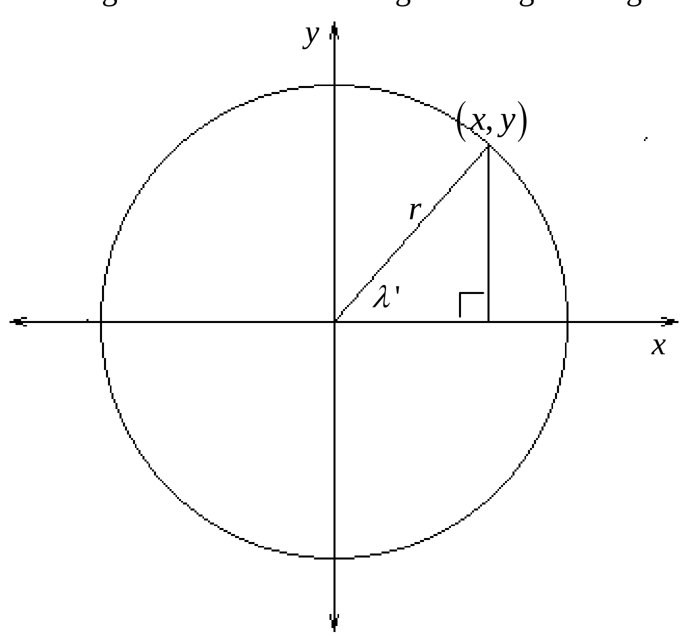
Consider angle λ below. In degrees, this angle measures $405°$. The reference angle, then is the acute angle formed between the positive *x*-axis and the terminal side of λ, which measures $45°$ because $405-360=45$.



We are now ready to define the six trigonometric functions without restricting the inputs. We start with sine.

> If θ is an angle in standard position and (x,y) is the point of intersection of the terminal side and the circle centered at the origin with radius r, then sine maps the measure of θ to y/r. We conflate the designation of the angle with its actual measurement and denote the sine function as $\sin(\theta)=y/r$.

At first glance, this definition seems to abandon the idea of inputs representing the measure of interior angles of right angles, that is, until we recall the concept of a reference angle. Recall angle λ from our diagram above whose measure was 405 degrees. Note that the intersection of the terminal side of λ with the circle centered at the origin with radius *r* is *by definition* the same intersection of the terminal side of λ' with the circle. Hence, sine maps any obtuse measure to the same output as the corresponding reference angle, which is by definition a positive acute angle and therefore can be imagined as the interior angle of a right triangle as sketched below.



Let θ be an angle in standard position. We denote the cosine function as $\cos(\theta)$, tangent as $\tan(\theta)$, cosecant as $\csc(\theta)$, secant as $\sec(\theta)$, and cotangent as $\cot(\theta)$. We define these functions as below.

> If θ is an angle in standard position and (x, y) is the point of intersection of the terminal side and the circle centered at the origin with radius r, then we define the six trigonometric functions as follows.
>
> $$\sin(\theta) = y/r$$
> $$\cos(\theta) = x/r$$
> $$\tan(\theta) = y/x$$
> $$\csc(\theta) = r/y$$
> $$\sec(\theta) = r/x$$
> $$\cot(\theta) = x/y$$

Now that we have the trigonometric functions defined, we will evaluate the functions for various real number inputs. Let's consider sine and cosine for the input value of 45-degrees or $\pi/4$ radians. The terminal side for a 45-degree angle in standard position lies on the line $y = x$. We need the intersection of this line with $x^2 + y^2 = r^2$, which describes the circle centered at the origin with radius *r*. Using substitution, we find the intersection below.

$$\begin{aligned} x^2 + y^2 &= r^2 & \qquad x^2 + y^2 &= r^2 \\ x^2 + x^2 &= r^2 & \qquad y^2 + y^2 &= r^2 \\ 2x^2 &= r^2 & \qquad 2y^2 &= r^2 \\ x^2 &= \frac{r^2}{2} & \qquad y^2 &= \frac{r^2}{2} \\ x &= \pm\frac{r}{\sqrt{2}} & \qquad y &= \pm\frac{r}{\sqrt{2}} \end{aligned}$$

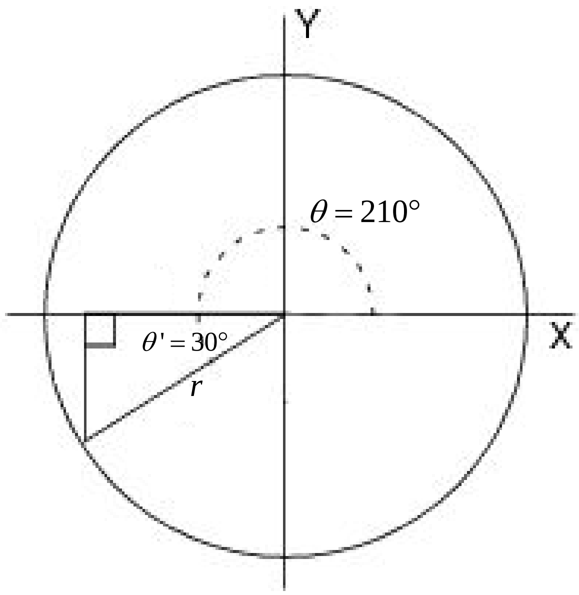
Since *r* represents the radius of the circle and is therefore positive, the points $\left(-r/\sqrt{2}, r/\sqrt{2}\right)$ and $\left(r/\sqrt{2}, -r/\sqrt{2}\right)$ are extraneous solutions while the actual points of intersection are $\left(r/\sqrt{2}, r/\sqrt{2}\right)$ and $\left(-r/\sqrt{2}, -r/\sqrt{2}\right)$. Remember that the terminal side of the 45-degree angle in standard position lies on the line $y = x$ only in the first quadrant. Therefore, $\left(r/\sqrt{2}, r/\sqrt{2}\right)$ is the solution of interest for our purposes. Using the definition, we can now evaluate sine and cosine at 45 degrees as follows.

$$\sin(45°) = \sin(\pi/4) = \frac{y}{r} = \frac{r/\sqrt{2}}{r} = \frac{r}{\sqrt{2}} \div r = \frac{\not{r}}{\sqrt{2}} \cdot \frac{1}{\not{r}} = \frac{1}{\sqrt{2}}$$
$$\cos(45°) = \cos(\pi/4) = \frac{x}{r} = \frac{r/\sqrt{2}}{r} = \frac{r}{\sqrt{2}} \div r = \frac{\not{r}}{\sqrt{2}} \cdot \frac{1}{\not{r}} = \frac{1}{\sqrt{2}}$$

What about the other point of intersection between $y = x$ and $x^2 + y^2 = r^2$? Consider the angle in standard position measuring $225°$ or $5\pi/4$. The terminal side of this angle lies on $y = x$ in the third quadrant so that its intersection with the circle is the point $\left(-r/\sqrt{2}, -r/\sqrt{2}\right)$. Hence, we can evaluate any one of the trigonometric functions at $5\pi/4$. For example, we will evaluate tangent.

$$\tan(225°) = \tan(5\pi/4) = \frac{y}{x} = \frac{-r/\sqrt{2}}{-r/\sqrt{2}} = 1$$

For most inputs, the trigonometric functions are more difficult to evaluate exactly, and we will rely on calculators to attain approximations. Nevertheless, as we see from the above discussion, calculating the exact value of a trigonometric function when the input is some multiple of $\pi/4$ is not difficult. The same can be said for multiples of $\pi/6$ or $30°$. Consider an angle of $210°$, which is coterminal with the angle measuring $-150°$. The reference angle is always positive, so the reference angle measures $30°$ as shown below.



From the intersection of the terminal side of $\theta = 210°$ and the circle centered at the origin with radius *r*, we drop a line segment perpendicular to the negative *x*-axis forming a right triangle as pictured above. From the 30º-60º-90º Rule, we know that the leg of the right triangle opposite the $30°$ angle has a length equal to one-half the length of the radius. Hence, the *y*-coordinate of the intersection point must be $-r/2$. Similarly, the other leg is $\sqrt{2}$ times longer. Hence, the *x*-

coordinate of the intersection point is $-r\sqrt{2}/2$. Using the intersection point $\left(-\frac{r}{2}, -\frac{r\sqrt{2}}{2}\right)$, we can evaluate any of the six trigonometric functions at $210°$. We evaluate secant below.

$$\sec(210°) = \sec(30°) = \sec(\pi/6) = \frac{r}{x} = \frac{r}{-r/2} = r \div \left(-\frac{r}{2}\right) = \cancel{r} \cdot \left(-\frac{2}{\cancel{r}}\right) = -2 .$$

Alert readers have probably noticed that each time we evaluate a trigonometric function the simplification process always involves dividing *r* into itself. Since *r* is an arbitrary radius, it eases the calculations to assign the radius to equal one. We call a circle with $r = 1$ a unit circle since its radius measures one unit. Since our definition of the six trigonometric functions relies on the intersection of the terminal side of an angle with a circle centered at the origin with *any* radius, we can simply use a radius of one. Hence, we can define the trigonometric functions equivalently as below.

If θ is an angle in standard position and (x, y) is the point of intersection of the terminal side and the *unit* circle centered at the origin, then we define the six trigonometric functions as follows.

$$\sin(\theta) = y$$
$$\cos(\theta) = x$$
$$\tan(\theta) = y/x$$
$$\csc(\theta) = 1/y$$
$$\sec(\theta) = 1/x$$
$$\cot(\theta) = x/y$$

Note that $x = \cos\theta$ and $y = \sin\theta$ by definition. Substituting into the definition of tangent, we obtain a fundamental identity.

$$\tan(\theta) = y/x$$
$$\tan(\theta) = \sin\theta/\cos\theta$$

In this way, we obtain the Fundamental Identities stated below.

Fundamental Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \ \cos\theta \neq 0$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}, \ \sin\theta \neq 0$$

Similarly, we also obtain the Reciprocal Identities stated here.

Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta}, \ \sin\theta \neq 0$$

$$\sin\theta = \frac{1}{\csc\theta}, \ \csc\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta}, \ \cos\theta \neq 0$$

$$\cos\theta = \frac{1}{\sec\theta}, \ \sec\theta \neq 0$$

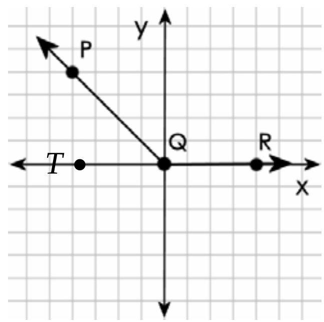
$$\cot\theta = \frac{1}{\tan\theta}, \ \tan\theta \neq 0$$

$$\tan\theta = \frac{1}{\cot\theta}, \ \cot\theta \neq 0$$

Example Exercise #1

Find the reference angle for an angle whose radian measure is $\frac{-13\pi}{4}$.

Recall that -2π or $-\frac{8\pi}{4}$ is a full rotation in the clockwise direction. Moreover, -3π or $-\frac{12\pi}{4}$ equals 1.5 rotations in the clockwise direction. Hence, $-\frac{13\pi}{4}$ equals 1.75 rotations in the clockwise direction, and $\angle RQP$ shown below has a radian measure of $-\frac{13\pi}{4}$.



The reference angle equals the acute angle between the terminal side and the *x*-axis and is always given a positive measure regardless of the direction of rotation. In this case, the reference angle equals $\angle TQP$. Its measure equals $\pi/4$.

Example Exercise #2

Find the exact value of $\cos\left(-\frac{13\pi}{4}\right)$.

We evaluate $\cos(\pi/4)$, but remember that the angle is actually in quadrant two where cosine is negative. We note that if the terminal side intersects the unit circle, then $\cos(\theta) = x$. Dropping a perpendicular from the intersection of the terminal side of the angle and the unit circle, we create a right triangle with a $45°$ interior angle. Using the Isosceles Right Triangle rule, we note that since the hypotenuse equals 1 unit (because the radius is one), then $x = 1/\sqrt{2}$ units. Hence, $\cos(\pi/4) = 1/\sqrt{2}$ and $\cos(-13\pi/4) = -1/\sqrt{2}$. If we rationalize, we have the answer below.

$$\cos\left(-\frac{13\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Suggested Homework

Section 5.2: #29-39 odd, #49-59 odd, #89
Section 5.4: #3-27 odd

Application Exercise

Sine and cosine appear in functions that model the motion of a spring. If a weight is at rest while hanging from a spring, then the spring is in a position of equilibrium. On a vertical number line, zero represents the equilibrium point. If the weight is set in motion with an initial velocity v_0 from location x_0, then the function below gives the vertical location at time *t* in relation to the point of equilibrium.

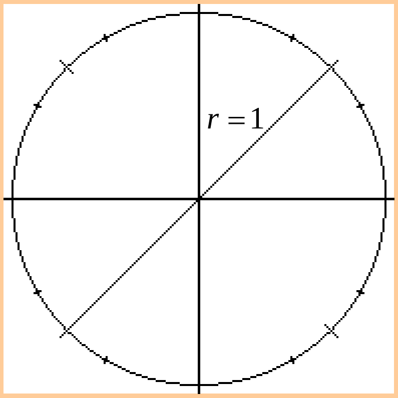
$$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$$

The letter ω represents a constant that depends on the stiffness of the spring and the amount of weight on the spring. Positive values of *x* indicate that the weight is below equilibrium while negative values of *x* indicate that the weight is above equilibrium.

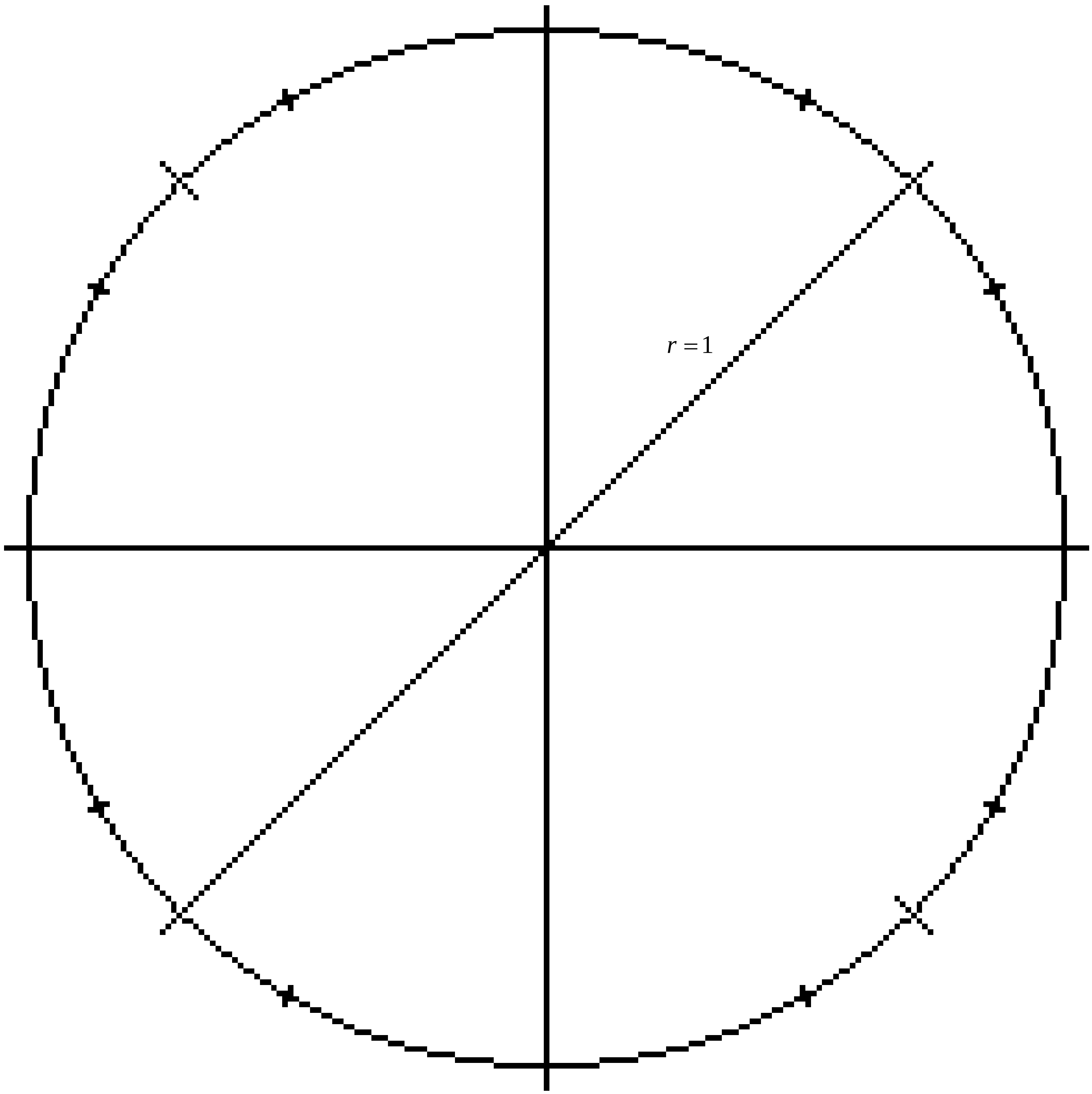
Since upward velocity is negative and locations below equilibrium are positive, use $v_0 = -3$, $x_0 = 2$ cm, and $\omega = \pi$ to locate the spring after 0.75 seconds.

Bonus Exercise

Use the 30º-60º-90º Rule and the 45º-45º-90º Rule from Geometry, to label a unit circle like the one pictured below with all the appropriate points of intersection for the terminal sides of angles in standard position measuring 0º, 30º, 45º, 60º, 90º, 120º, 135º, 150º, 180º, 210º, 225º, 240º, 270º, 300º, 315º, & 330º.



Please do not try to label this particularly tiny unit circle but instead draw your own larger circle centered at the origin and assign the radius to equal one or use the circle on the following page. *Show your work in the form of all the appropriate triangles formed by dropping perpendiculars from the points of intersection to the x-axis.*


r = 1