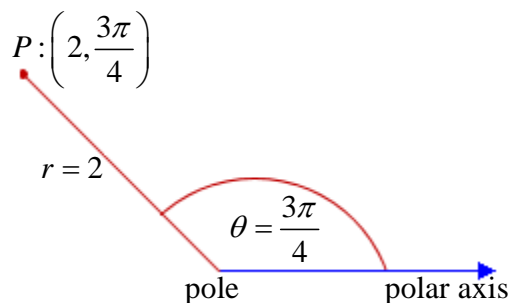


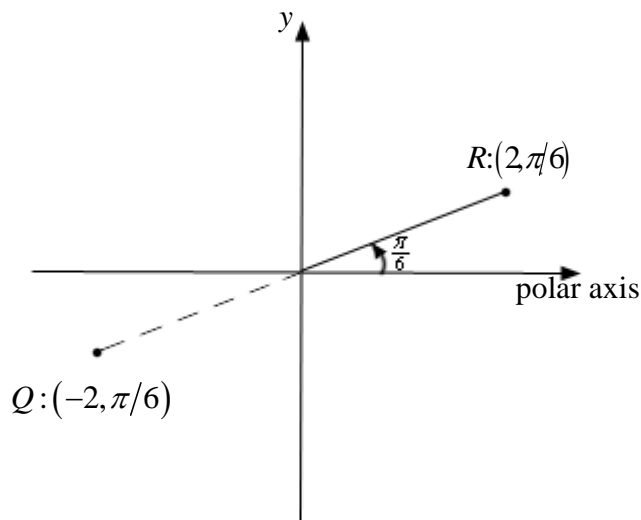
Polar Equations

In previous lectures, we have used the rectangular coordinate system, which employs an ordered pair (x, y) to represent a point in a plane. The two numbers, the abscissa and the ordinate, respectively, represent the distances between the point and two perpendicular axes (the x and y -axes). Polar coordinates represent another way to represent points in a plane. With polar graphing, we need only one axis called the *polar axis* and a point on it called the *pole*. These two constructs correspond to the x -axis and the origin in the rectangular polar system. Sometimes we include the y -axis when we represent the polar coordinate system, but we do so only to help orient ourselves after having accustomed ourselves to the rectangular coordinate system.

With polar coordinates, we still employ an ordered pair to represent a point in a plane, but only one of the numbers represents a linear distance. When the first number is positive, we imagine the point as a point on the terminal side of an angle in standard position. The first number in the ordered pair is r , the distance from the pole to the point. The second number in the ordered pair is θ , the measure of the angle on whose terminal side the point lies.



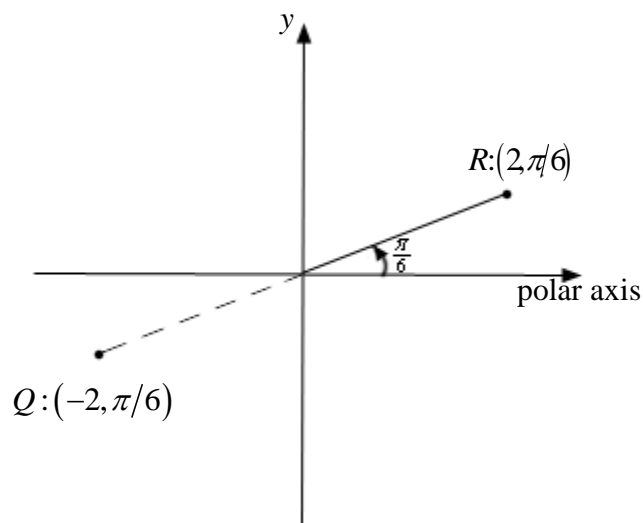
As we have discovered, an angle in standard position has an infinite number of coterminal angles. Hence, an infinite number of ordered pairs can represent a single point in the polar coordinate system just by using coterminal measures for θ . Worse, we can also imagine points as laying on the ray opposite the terminal side of θ , in which case, $r < 0$, as shown with the point Q below.



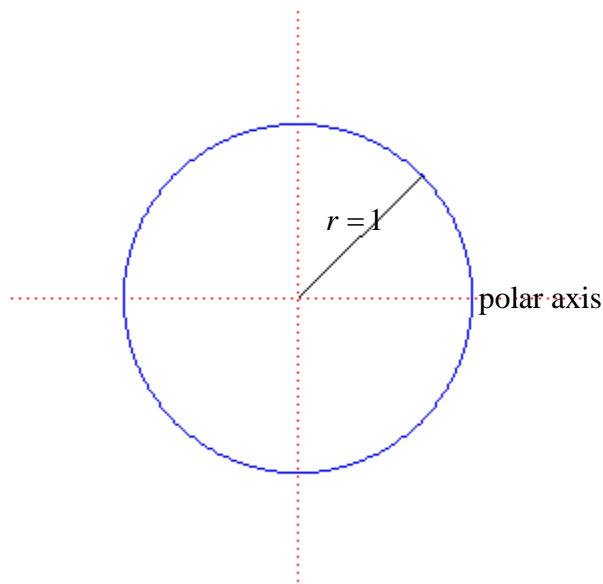
To plot a point in polar, requires two basic steps. First, look at θ and orient in the correct direction. Second, travel exactly r -units from the pole in the oriented direction. If $r < 0$, go in the opposition direction. It is important to note that a point any particular quadrant does not

necessarily have a particular sign. For instance, all of the following coordinates describe point R : $(2, \pi/6)$, $(-2, 7\pi/6)$, $(2, -11\pi/6)$, and $(-2, -5\pi/6)$. Besides, the idea of quadrants is a holdover from rectangular coordinates anyway since we really only have one axis in the polar coordinate system.

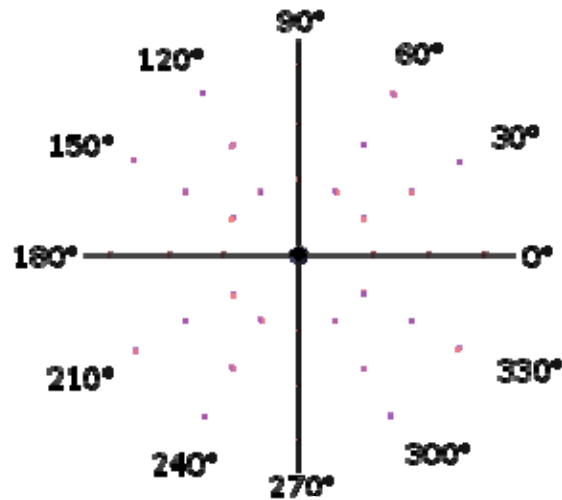
In the rectangular coordinate system, $y = b$ represents a horizontal line passing through b on the y -axis. The polar coordinate system has its own analog to this constant function because the equation $\theta = \lambda$ represents a line passing through the pole with a direction equal to λ . For instance, we can use the equations $\theta = \pi/6$ or $\theta = 30^\circ$ or $\theta = 7\pi/6$ to describe the line containing the pole and points R and Q below.



Similarly, $x = a$ represents a vertical line passing through a on the x -axis in the rectangular coordinate system, and this relation has an analog in the polar system because the equation $r = a$ represents a circle centered at the pole with a radius a .



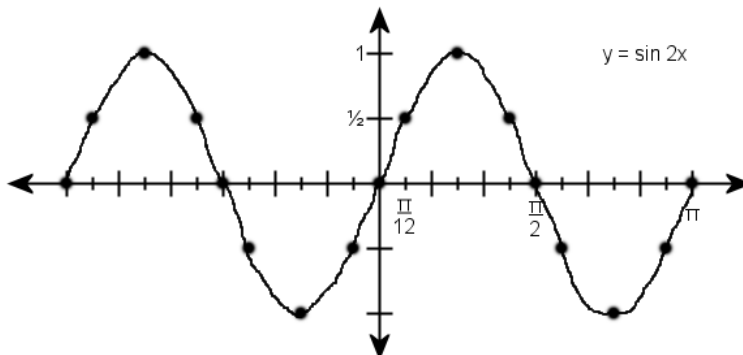
To aid in graphing points and relations, it is common to create polar grids using circles as analogs to tick marks on the axes in the rectangular system as shown below.



Just as we first learn to graph relations in the rectangular coordinate system using point-by-point plotting, so can we graph relations in the polar system. Consider a function in the form $r = f(\theta)$. The following items are helpful for generating the graph of the function in the polar plane.

1. Look for symmetry and know that there are many common graphs such as circles, cardioids, limaçons, lemniscates, spirals, and rhodonea curves (see the function gallery in section 7.6 of the textbook).
2. Identify the values of θ that maximize and minimize r .
3. Identify the zeros of the function, i.e., find $\theta \ni r = 0$. These will be angle values where the curve passes through the pole.
4. When in doubt about the shape of the graph, methodically plot points for incremental increases in θ .
5. Plot points over a full cycle of the un-dilated function.
6. Watch for the graph to repeat itself, tracing out the same shape again periodically.

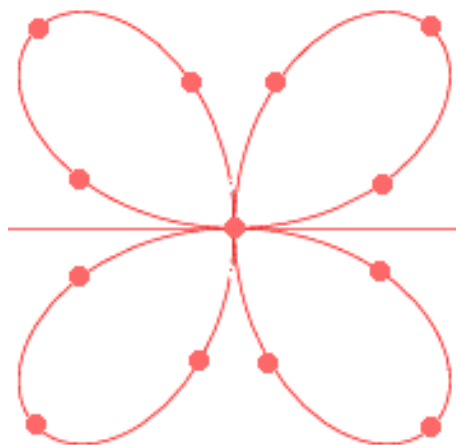
Let's graph $r = \sin 2\theta$. Looking at the graph of $y = \sin 2x$ in the rectangular coordinate system given on the following page, we can read values of r and θ if we treat the x -values as angles and y -values are radii.



The interval $I = [-\pi, \pi]$ represents a full cycle of $y = \sin x$. The graph of $y = \sin 2x$ passes through the x -axis at $x = -\pi, x = -\pi/2, x = 0, x = \pi/2,$ and $x = \pi$. Treating the x -values as angles and y -values are radii, we see that the curve in polar form will have a radius of zero (meaning it will pass through the pole) when θ equals $-\pi, -\pi/2, 0, \pi/2,$ and π . The graph in rectangular form takes a minimum value of -1 when $x = -\pi/4$ and $x = 3\pi/4$. Likewise, $y = \sin 2x$ has maximum value of 1 when $x = -3\pi/4$ and $x = \pi/4$. Now, we generate a table for increasing θ -values making sure to include these key x -values as key θ -values.

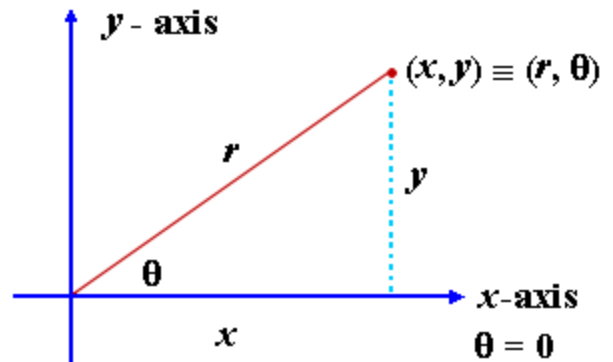
θ	$-\pi$	$-\frac{11\pi}{12}$	$-\frac{3\pi}{4}$	$-\frac{7\pi}{12}$	$-\frac{\pi}{2}$	$-\frac{5\pi}{12}$	$-\frac{\pi}{4}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{3\pi}{4}$	$\frac{11\pi}{12}$	π
r	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0

Plotting these points, the graph below takes shape as shown. Plotting further values of the function for greater values of θ will show the graph repeating itself. We call this curve the quadrifolium. It belongs to a family of curves called rhodonea curves, sometimes referred to as a rose curve.



We can put our familiarity with the rectangular equations to use by converting polar equations to rectangular form or vice versa. When we convert from one system to the other, we recall the relationships established in our definitions of the trigonometric functions, namely, the following.

$$x^2 + y^2 = r^2, x = r \cos \theta, \text{ and } y = r \sin \theta$$



For example, consider the rectangular representation $y = x^2$. We start by substituting for y and x as below.

$$y = x^2$$

$$r \sin \theta = (r \cos \theta)^2$$

$$r \sin \theta = r^2 \cos^2 \theta$$

Dividing both sides by r , we obtain the following.

$$r \sin \theta = r^2 \cos^2 \theta$$

$$\sin \theta = r \cos^2 \theta \quad \text{or} \quad r = 0$$

(If the reader is wondering where the equation $r = 0$ came from, just remember that only Chuck Norris can divide by zero.) Dividing by $\cos^2 \theta$, we continue converting below using basic identities.

$$\sin \theta = r \cos^2 \theta$$

$$\frac{\sin \theta}{\cos^2 \theta} = r$$

$$r = \frac{1}{\cos \theta} \frac{\sin \theta}{\cos \theta}$$

$$r = \sec \theta \tan \theta$$

Since $r = 0$, represents the pole, which is included in the graph of $r = \sec^2 \theta \sin \theta$ when $\theta = 0$, we may drop $r = 0$, and simply represent $y = x^2$ as $r = \sec \theta \tan \theta$.

Let's consider converting $r = 1 - \cos \theta$ to rectangular representation. Considering the relationship $x^2 + y^2 = r^2$, we multiply the equation by r as follows.

$$\begin{aligned} r &= 1 - \cos \theta \\ r^2 &= r \cdot (1 - \cos \theta) \\ r^2 &= r - r \cos \theta \\ r^2 + r \cos \theta &= r \\ r &= r^2 + r \cos \theta \end{aligned}$$

Next, we square both sides and obtain the following.

$$\begin{aligned} r &= r^2 + r \cos \theta \\ r^2 &= (r^2 + r \cos \theta)^2 \end{aligned}$$

Now, we conclude as below using substitution.

$$\begin{aligned} r^2 &= (r^2 + r \cos \theta)^2 \\ x^2 + y^2 &= (x^2 + y^2 + x)^2 \end{aligned}$$

Thus, we have converted the relationship from polar representation to rectangular representation.

This process of conversion sometimes has the possibility of introducing extraneous roots as above both when we squared both sides and when we multiplied the equation by the variable r . To check, we consider both steps. For instance, we multiplied both sides by r to go from $r = 1 - \cos \theta$ to $r = r^2 + r \cos \theta$, but in this case no extraneous roots are introduced because $r = 0$ on the graph of $r = 1 - \cos \theta$ when $\theta = \pi$. Since the graph of $r = 1 - \cos \theta$ passes through the pole, then changing the equation to $r = r^2 + r \cos \theta$ does not introduce a root not already present. Squaring both sides of an equation can also introduce extraneous roots. When we squared both sides of the equation, we transformed $r = r^2 + r \cos \theta$ into $r = \pm(r^2 + r \cos \theta)$. We have already established above that $r^2 + r \cos \theta = 1 - \cos \theta$. Moreover, the graph of $r = 1 - \cos \theta$ is equivalent to the graph of $r = -1 - \cos \theta$. Hence, squaring did not introduce any extraneous roots.

Comparing the polar representation $r = 1 - \cos \theta$ to the rectangular representation $x^2 + y^2 = (x^2 + y^2 + x)^2$, we begin to see how polar has certain advantages (in this particular case) since graphing $x^2 + y^2 = (x^2 + y^2 + x)^2$ does not look like a simple task. Indeed, there was a time when the polar coordinate system competed so to speak with the rectangular system for the dominant method for representing relations in a plane. In large part, the rectangular system won, but polar still has its particular demesne where it holds sway.

Suggested Homework

Section 7.6: #9-11 odd, #27, #29, #55-69 odd, #75-81 odd, #87-95 odd

Application Exercise

Show that a polar equation for the straight line $y = mx$ is $\theta = \tan^{-1} m$.