Powers and Roots of Complex Numbers

In this lesson, we present a theorem without proof then use the theorem to find all the roots (real or non-real) of a polynomial equation. We start with the definition below.

The complex number $w = a + bi$ is an nth root of the complex number z if $(a+bi)^n = z$.

Now, we state the theorem.

For any positive integer *n*, then w is an *n*th root of the complex number $z = r(\cos\theta + i\sin\theta)$ if

$$w = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

Using this theorem, we can solve the equation $x^6 - 1 = 0$ as below.

$$x^6 - 1 = 0$$
$$x^6 = 1$$
$$x = \sqrt[6]{1}$$

From the above work, we see that solving $x^6 - 1 = 0$ requires finding the six sixth-roots of 1. To apply our theorem, we first note that $1 = 1(\cos 0 + i\sin 0)$. Then, we substitute into the expression $w = 1^{\frac{1}{6}}\left[\cos\left(\frac{0 + 2k\pi}{6}\right) + i\sin\left(\frac{0 + 2k\pi}{6}\right)\right]$ for the values of $k = 0, 1, 2, \ldots, 5$. For $k = 0$, we obtain,

$$w_0 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0 + 2\cdot 0\cdot\pi}{6}\right) + i\sin\left(\frac{0 + 2\cdot 0\cdot\pi}{6}\right)\right] = 1\left[\cos\left(\frac{0}{6}\right) + i\sin\left(\frac{0}{6}\right)\right] = 1.$$

For $k = 1$,

$$w_1 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0 + 2\cdot 1\cdot\pi}{6}\right) + i\sin\left(\frac{0 + 2\cdot 1\cdot\pi}{6}\right)\right] = 1\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

For $k = 2$,

$$w_2 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0+2\cdot 2\cdot \pi}{6}\right)+i\sin\left(\frac{0+2\cdot 2\cdot \pi}{6}\right)\right]=1\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]=-\frac{1}{2}+i\frac{\sqrt{3}}{2}.$$

For $k=3$,

$$w_3 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0+2\cdot 3\cdot \pi}{6}\right)+i\sin\left(\frac{0+2\cdot 3\cdot \pi}{6}\right)\right]=1\left[\cos(\pi)+i\sin(\pi)\right]=-1.$$

For $k=4$,

$$w_4 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0+2\cdot 4\cdot \pi}{6}\right)+i\sin\left(\frac{0+2\cdot 4\cdot \pi}{6}\right)\right]=1\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right]=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

Finally, for $k=5$,

$$w_5 = 1^{\frac{1}{6}}\left[\cos\left(\frac{0+2\cdot 5\cdot \pi}{6}\right)+i\sin\left(\frac{0+2\cdot 5\cdot \pi}{6}\right)\right]=1\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right]=\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

Hence, the six sixth-roots of 1 are 1, -1, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, and $\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

Suggested Homework

Section 7.5: #41, #42, #43, #44, #51, #52

Application Exercise

Find all fourth roots of -16.