

Complex Numbers

When we study algebra, we discover that not every quadratic equation has a solution set of real numbers. For example, $x^2 + 1 = 0$ has no real-number solution since there is no real number x such that $x = \pm\sqrt{-1}$. In this lecture, we will construct a set of numbers in which every quadratic equation has a solution starting with the definition below.

$$\sqrt{-1} = i$$

Thus, we define i to be the number whose square is negative one, that is, $i^2 = -1$. Hence, $\pm i$ are solutions for $x^2 + 1 = 0$.

To continue our construction to include solutions for equations of the form $x^2 + a = 0$ where $a > 0$, we state the definition below.

$$\text{If } a \geq 0, \text{ then } \sqrt{-a} = \sqrt{a} \cdot i.$$

We call the numbers defined above *imaginary numbers*.

Let b be a real number. Then, we call bi an *imaginary number*.

If we add an imaginary number to a real number, we define the result to be a *complex number*.

Let a and b be real numbers. Then, we call $a + bi$ a *complex number*.

Note that the definition above allows a and b to be zero. Hence, any imaginary number like $5i$ is a complex number, but the converse is not true since $2 + 5i$ is complex but not imaginary. Moreover, any real number a is complex since $a = a + 0i$, but, again, the converse is not true since $2 + 5i$ is complex but not real. All this leads us to the definition below.

Let a and b be real numbers. Then, we call $a + bi$ a *complex number*. If $b \neq 0$, we call $a + bi$ a *non-real complex number*. If $a = 0$, we call $a + bi$ an *imaginary number*. If $a \neq 0 \wedge b = 0$, we call $a + bi$ a *real number*.

We call the form $a + bi$ the *standard form* of a complex number.

We define equality among complex numbers as below.

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

We define addition and subtraction as follows.

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) - (c + di) &= (a - c) + (b - d)i\end{aligned}$$

We define multiplication and division as below.

Let $c + di$ be a non-zero complex number. Then,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

and

$$\begin{aligned} (a + bi) \div (c + di) &= (a + bi) \cdot \frac{1}{c + di} \\ &= (a + bi) \cdot \frac{1}{c + di} \cdot \frac{c - di}{c - di} \\ &= (a + bi) \cdot \frac{c - di}{c^2 + d^2} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 + d^2} \\ &= \frac{(ac + bd) - (ad - bc)i}{c^2 + d^2}. \end{aligned}$$

Note that the definition for division relies on the interesting fact that the product of a complex number and its conjugate is real.

There exists a one-to-one correspondence between the set of real numbers and the set of imaginary numbers. Moreover, these two sets have an element in common, namely, $0 = 0i$. This one-to-one correspondence and the common element allow us to graph the set of complex numbers in a coordinate plane.

For the coordinate system, the horizontal axis is the real axis while the vertical axis is the imaginary axis. We plot real numbers as points on the horizontal axis and imaginary numbers as points on the vertical axis. Non-real, non-imaginary complex numbers fall into one of the four quadrants. Note the complex numbers 3 , $-4i$, and $-2 + 4i$ in Figure 1 below.

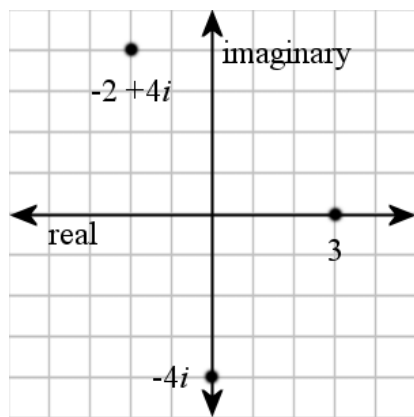


Figure 1

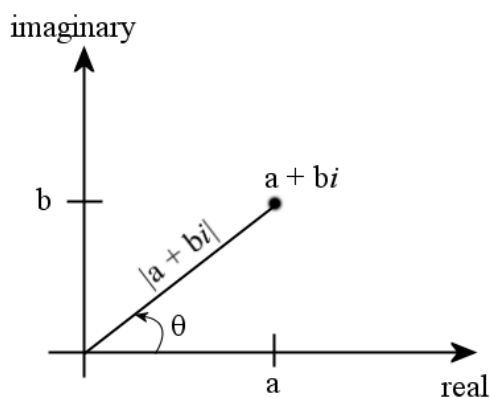
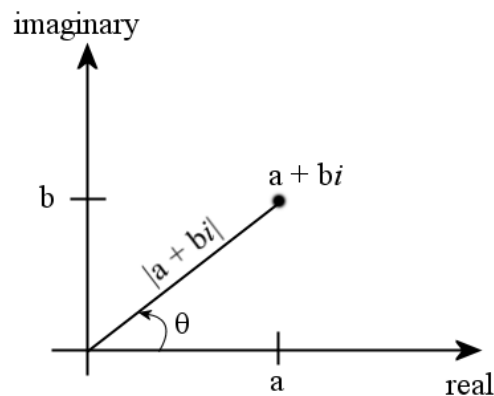


Figure 2

We sometimes use a geometric vector with the initial point at the origin and terminal point (a, b) to represent a complex number as shown in Figure 2 above. The magnitude of the vector equals the absolute value or *modulus* of the complex number.

Let $a + bi$ be a complex number. Let $r = |a + bi|$. Then, we call r the *modulus* of $a + bi$, and $r = \sqrt{a^2 + b^2}$.

Using the geometric vector representation of a complex number, we can derive a form for complex numbers called the trigonometric form. Looking at the geometric representation below, we see that $a = r \cos \theta$ and $b = r \sin \theta$.



Substituting into the standard form we obtain the following.

$$a + bi = r \cos \theta + r \sin \theta \cdot i = r(\cos \theta + i \sin \theta)$$

Hence, we have the definition below.

Let $z = a + bi$ be a complex number. Then, the *trigonometric form* of z is $z = r(\cos \theta + i \sin \theta)$ where $r = \sqrt{a^2 + b^2}$ and θ is an angle in standard position whose terminal side contains the point (a, b) . We call r the *modulus* and θ the *argument* of z .

The trigonometric form simplifies the process of finding the product of two complex numbers according to the theorem below.

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then,

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

For proof, we multiply and employ Sum Identities as below.

$$z_1 z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) \cdot r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$z_1 z_2 = r_1 r_2 \cos \theta_1 \cos \theta_2 + \cos \theta_1 i \sin \theta_2 + \cos \theta_2 i \sin \theta_1 + i^2 \sin \theta_1 \sin \theta_2$$

$$z_1 z_2 = r_1 r_2 \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + \cos \theta_1 i \sin \theta_2 + \cos \theta_2 i \sin \theta_1$$

$$z_1 z_2 = r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1)]$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

The trigonometric form also simplifies the process for finding the quotient of two complex numbers according to the theorem below whose proof so similar that it is left as an exercise.

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \neq 0$. Then,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Suggested Homework

Section 7.4: #1-7 odd, #13-19 odd, #29-33 odd, #41-53 odd

Application Exercise

Powers of i possess an interesting periodic property. Since $i^2 = -1$, we obtain the following pattern.

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

$$i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$$

$$i^7 = i^4 \cdot i^3 = 1 \cdot -i = -i$$

$$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$$

⋮

Thus, every natural number power of i can be expressed as one of the numbers i , -1 , $-i$, or 1 . Use this fact and the pattern above to simplify i^{228} .