

Vectors

We use the term vector to refer to a finite list of numbers. If the list includes just two numbers, an ordered pair, we call the list a vector in \mathbb{R}^2 . In this course, we will deal exclusively with vectors in \mathbb{R}^2 and commonly call them two-space vectors or just simply vectors.

Let $z_1 \wedge z_2$ be real numbers. Then, \vec{u} , \mathbf{v} , \mathbf{w} , \mathbf{z} , \overline{OA} , and $\mathbf{0}$ below are all examples of vectors and accepted notations for vectors.

$$\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathbf{v} = \langle -1, 4 \rangle, \mathbf{w} = (0.2, 0.3), \mathbf{z} = \langle z_1, z_2 \rangle, \overline{OA} = (2, 3), \text{ and } \mathbf{0} = \langle 0, 0 \rangle$$

Two \mathbb{R}^2 vectors are equal if and only if their corresponding entries are equal. Hence, if $\mathbf{z} = \mathbf{v}$ from the examples above, then $z_1 = -1$ and $z_2 = 4$. Given two \mathbb{R}^2 vectors like \mathbf{u} and \mathbf{v} , their sum is the vector $\mathbf{u} + \mathbf{v}$ whose entries are the sums of the corresponding entries of \mathbf{u} and \mathbf{v} . Let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -1, 4 \rangle$. Then, $\mathbf{u} + \mathbf{v} = \langle 3 + (-1), -2 + 4 \rangle = \langle 2, 2 \rangle$. Given a vector \mathbf{u} and a real number c , then $c\mathbf{u}$ is a *scalar multiple* of \mathbf{u} obtained by multiplying each entry in \mathbf{u} by c . For instance, if $\mathbf{u} = \langle 3, -2 \rangle$ and $c = 7$, then $c\mathbf{u} = 7\langle 3, -2 \rangle = \langle 7(3), 7(-2) \rangle = \langle 21, -14 \rangle$.

In summary then, we have the following definitions.

A *two-space vector* is an ordered pair of real numbers. If both numbers are zero, we call the vector the *zero vector*.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be two-space vectors and let k be any real number. Then, $k\mathbf{u} = \langle ku_1, ku_2 \rangle$, $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$, and $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$. We call $k\mathbf{u}$ a *scalar multiple* of \mathbf{u} . We call $\mathbf{u} + \mathbf{v}$ the *sum* (or the *resultant*) of the vectors \mathbf{u} and \mathbf{v} . We call $\mathbf{u} - \mathbf{v}$ the *difference* of the vectors \mathbf{u} and \mathbf{v} .

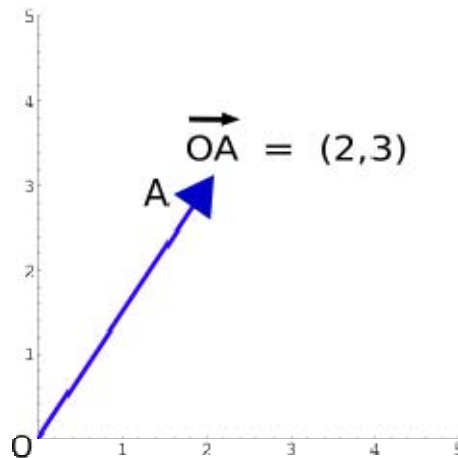
Geometric Representation of Vectors

Consider a rectangular coordinate system in the plane. Since every point in the plane is determined by an ordered pair, we can identify a geometric point (a, b) with a vector $\langle a, b \rangle$.

Hence, we may regard \mathbb{R}^2 as the set of all vectors with two real number entries just as we regard \mathbb{R}^2 as the set of all points in the plane.

The geometric visualization of a nonzero vector is a directed line segment from the origin to the point. We aid this visualization by including an arrow at the terminal end of the segment directing our attention to the point that *is* the vector. It turns out, that for some applications, we can visualize this directed line segment as free of its standard position in the plane and just pick

it up and place it elsewhere in the plane. Nevertheless, we will adhere to directed line segments in standard position exemplified by vector \overrightarrow{OA} below.



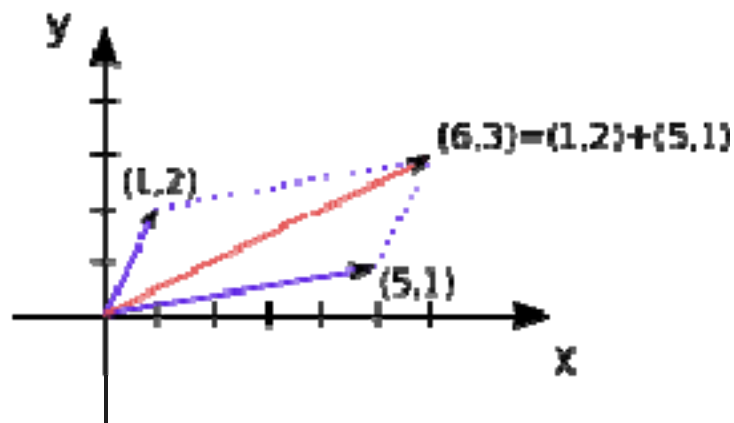
We call the angle that the vector makes with the x -axis its *direction*. We call the length of the vector its *magnitude*. We denote magnitude with absolute value. Hence, $|\overrightarrow{OA}|$ refers to the magnitude of the vector, which we obtain from the Pythagorean Theorem.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a two-space vector. We call $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$ the magnitude of \mathbf{u} .

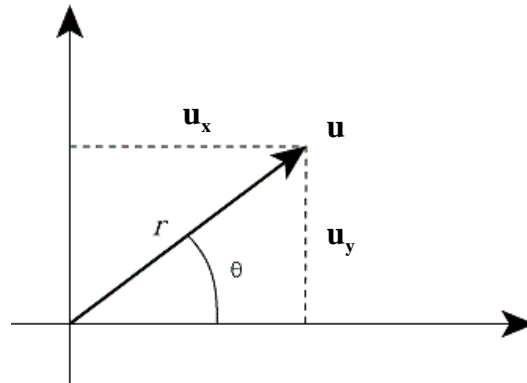
The sum of two vectors has a useful geometric representation summarized by the *Parallelogram Rule* stated below.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be two-space vectors. Then, the resultant $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are \mathbf{u} , \mathbf{v} , and $\mathbf{0}$.

The following figure demonstrates the rule in action.



Considering the Parallelogram Rule, we can view any nonzero vector as the sum or resultant of two special vectors called the horizontal component and the vertical component. If \mathbf{u} is a vector with magnitude r and direction θ , we denote the horizontal component of \mathbf{u} as \mathbf{u}_x and the vertical component of \mathbf{u} as \mathbf{u}_y . The figure below provides an illustration.



Using the figure above and the trigonometric ratios, we obtain the following equations.

$$\cos \theta = \frac{|\mathbf{u}_x|}{r} \Rightarrow |\mathbf{u}_x| = r \cos \theta$$

$$\sin \theta = \frac{|\mathbf{u}_y|}{r} \Rightarrow |\mathbf{u}_y| = r \sin \theta$$

If the direction of \mathbf{u} is negative, then we can write the following.

$$|\mathbf{u}_x| = |r \cos \theta|$$

$$|\mathbf{u}_y| = |r \sin \theta|$$

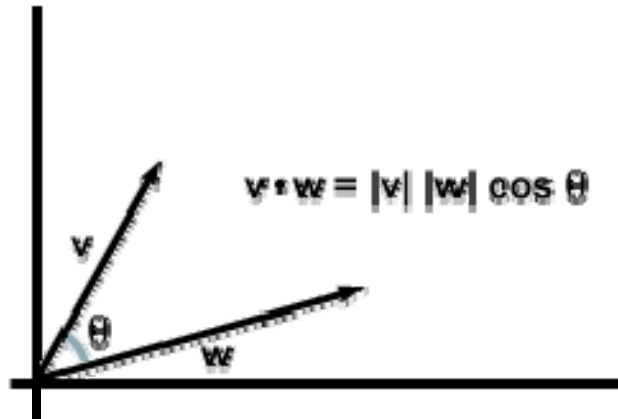
Above, we defined the sum and difference of two vectors. We also defined *scalar multiplication*. Intuitive readers may wonder about *vector multiplication*. Actually, there is more than one definition for multiplying two vectors. We are interested in the *dot product*, defined below.

The dot product of two nonzero vectors \mathbf{v} and \mathbf{w} is the number

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos \theta$$

where θ is the angle between \mathbf{v} and \mathbf{w} , $0 \leq \theta \leq \pi$. If either \mathbf{v} or \mathbf{w} is $\mathbf{0}$, we define $\mathbf{v} \cdot \mathbf{w} = 0$.

The following figure illustrates the dot product.



If the angle between \mathbf{v} and \mathbf{w} is 90° , then the dot product of \mathbf{v} and \mathbf{w} will be zero because $\cos 90^\circ = 0$. Since any nonzero vector has a positive magnitude, the only way the dot product of two nonzero vectors can equal zero is if the angle between the two vectors is a right angle. Hence, we have the theorem below.

Two nonzero vectors \mathbf{u} and \mathbf{v} are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Linear Combinations

We call any vector with a magnitude of one unit a *unit vector*. Two special unit vectors are the vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. We can represent any vector $\mathbf{u} = \langle u_1, u_2 \rangle$ using \mathbf{i} and \mathbf{j} as below.

$$\mathbf{u} = \langle u_1, u_2 \rangle = u_1 \langle 1, 0 \rangle + u_2 \langle 0, 1 \rangle = u_1 \mathbf{i} + u_2 \mathbf{j}$$

We call the form $u_1 \mathbf{i} + u_2 \mathbf{j}$ a *linear combination*.

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and given scalars c_1, c_2, \dots, c_n , the vector \mathbf{z} given by

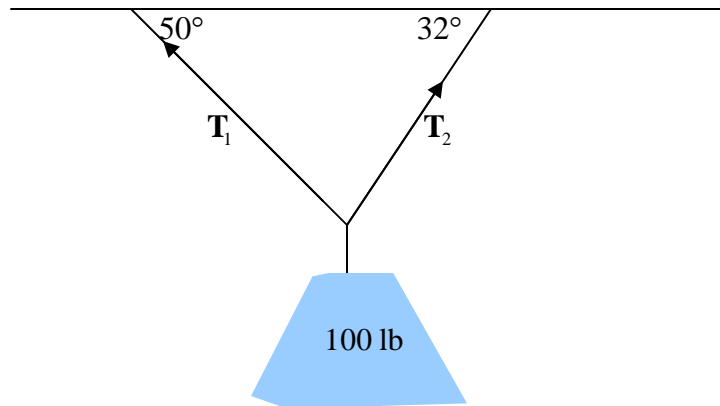
$$\mathbf{z} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n$$

is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with scalars c_1, c_2, \dots, c_n .

Applications of Vectors

We can use vectors to represent a force because a force has both magnitude and direction. If several forces are acting on an object, the resultant force experienced by the object is the vector sum of these forces.

Consider a 100-lb weight hanging from a horizontal surface by two wires as shown in the diagram below.



We can find the magnitude of the tensions \mathbf{T}_1 and \mathbf{T}_2 in both wires. We first express \mathbf{T}_1 and \mathbf{T}_2 in terms of their horizontal and vertical components as below.

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos(50^\circ)\mathbf{i} + |\mathbf{T}_1|\sin(50^\circ)\mathbf{j}$$

$$\mathbf{T}_2 = |\mathbf{T}_2|\cos(32^\circ)\mathbf{i} + |\mathbf{T}_2|\sin(32^\circ)\mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight \mathbf{w} , so we have the following equalities below.

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}$$

$$(-|\mathbf{T}_1|\cos(50^\circ)\mathbf{i} + |\mathbf{T}_1|\sin(50^\circ)\mathbf{j}) + (|\mathbf{T}_2|\cos(32^\circ)\mathbf{i} + |\mathbf{T}_2|\sin(32^\circ)\mathbf{j}) = 100\mathbf{j}$$

$$-|\mathbf{T}_1|\cos(50^\circ)\mathbf{i} + |\mathbf{T}_2|\cos(32^\circ)\mathbf{i} + |\mathbf{T}_1|\sin(50^\circ)\mathbf{j} + |\mathbf{T}_2|\sin(32^\circ)\mathbf{j} = 100\mathbf{j}$$

$$(-|\mathbf{T}_1|\cos(50^\circ) + |\mathbf{T}_2|\cos(32^\circ))\mathbf{i} + (|\mathbf{T}_1|\sin(50^\circ) + |\mathbf{T}_2|\sin(32^\circ))\mathbf{j} = 0\mathbf{i} + 100\mathbf{j}$$

Thus, we have the system of equations below.

$$-|\mathbf{T}_1|\cos(50^\circ) + |\mathbf{T}_2|\cos(32^\circ) = 0$$

$$|\mathbf{T}_1|\sin(50^\circ) + |\mathbf{T}_2|\sin(32^\circ) = 100$$

Next, we solve the first of these equations for $|\mathbf{T}_2|$ as below.

$$\begin{aligned}
-|\mathbf{T}_1|\cos(50^\circ) + |\mathbf{T}_2|\cos(32^\circ) &= 0 \\
|\mathbf{T}_2|\cos(32^\circ) &= |\mathbf{T}_1|\cos(50^\circ) \\
|\mathbf{T}_2| &= \frac{|\mathbf{T}_1|\cos(50^\circ)}{\cos(32^\circ)}
\end{aligned}$$

Substituting for $|\mathbf{T}_2|$ into the second equation, we obtain the following.

$$\begin{aligned}
|\mathbf{T}_1|\sin(50^\circ) + |\mathbf{T}_2|\sin(32^\circ) &= 100 \\
|\mathbf{T}_1|\sin(50^\circ) + \frac{|\mathbf{T}_1|\cos(50^\circ)}{\cos(32^\circ)}\sin(32^\circ) &= 100 \\
|\mathbf{T}_1|(\sin(50^\circ) + \cos(50^\circ)\tan(32^\circ)) &= 100 \\
|\mathbf{T}_1| &= \frac{100}{\sin(50^\circ) + \cos(50^\circ)\tan(32^\circ)} \\
|\mathbf{T}_1| &\approx 85.64 \text{ lb}
\end{aligned}$$

Returning to our expression for $|\mathbf{T}_2|$, we conclude as below.

$$\begin{aligned}
|\mathbf{T}_2| &= \frac{|\mathbf{T}_1|\cos(50^\circ)}{\cos(32^\circ)} \\
|\mathbf{T}_2| &\approx 85.64 \text{ lb} \frac{\cos(50^\circ)}{\cos(32^\circ)} \\
|\mathbf{T}_2| &\approx 64.91 \text{ lb}
\end{aligned}$$

Suggested Homework

Section 7.3: #13-29 odd, #37-69 odd

Application Exercise

Find the amount of force required to push an 800-pound block of ice up a ramp that is inclined 10 degrees. Hint: The result of the 800-pound force and the force of the push is perpendicular to the incline.