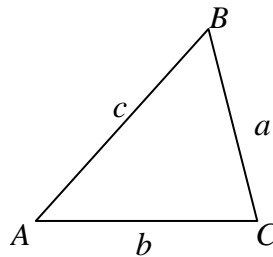


Law of Cosines

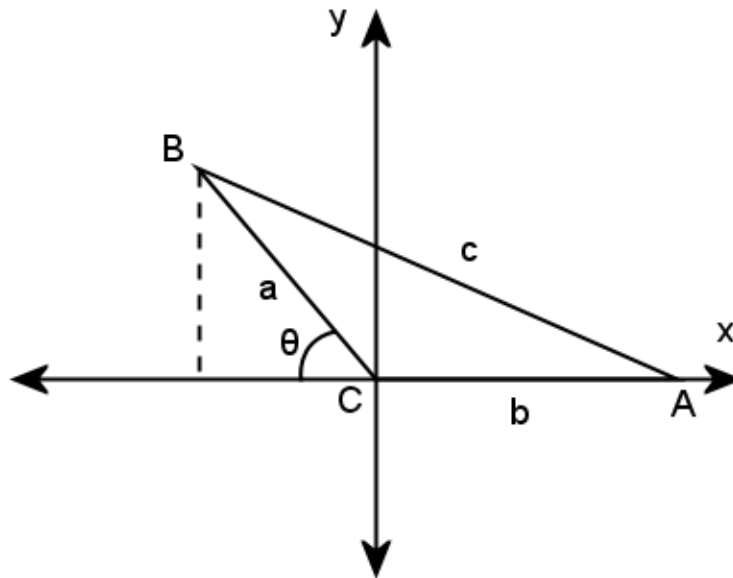
In the previous lecture, we used the Law of Sines to solve triangles given two angle measures (AAS and ASA cases) or two side measures and the un-included angle measures (the ambiguous SSA case). In this lecture, we use the Law of Cosines stated below to solve triangles given three side measures (SSS case) or given two side measures with the included angle's measure (SAS case).

The *Law of Cosines* states that given any oblique triangle $\triangle ABC$ or any right triangle ABC where c is the hypotenuse, then we have the following equality.

$$c^2 = a^2 + b^2 - 2ab \cos C$$



For proof, we consider a general triangle. Without loss of generality, we have one vertex on the origin and another on the x -axis.



Let the vertex at $(0,0)$ be C (and let C denote the vertex, the interior angle at the vertex, and the angle measure). Let A be on the x -axis. Note that $A = (b,0)$ and $B = (a \cdot \cos C, a \cdot \sin C)$. Calculating the distance between A and B using the distance formula, we have the following.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$c = \sqrt{(a \cdot \cos C - b)^2 + (a \cdot \sin C - 0)^2}$$

$$c = \sqrt{(a \cdot \cos C - b)^2 + (a \cdot \sin C)^2}$$

Next, we square both sides as below.

$$(c)^2 = \left(\sqrt{(a \cdot \cos C - b)^2 + (a \cdot \sin C)^2} \right)^2$$

$$c^2 = (a \cdot \cos C - b)^2 + (a \cdot \sin C)^2$$

$$c^2 = (a \cdot \cos C - b)(a \cdot \cos C - b) + a^2 \sin^2 C$$

$$c^2 = a^2 \cos^2 C - ab \cos C - ab \cos C + b^2 + a^2 \sin^2 C$$

$$c^2 = a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C$$

Finally, we conclude using the commutative property of addition and the Pythagorean Identity as follows.

$$c^2 = a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C$$

$$c^2 = a^2 \cos^2 C + a^2 \sin^2 C + b^2 - 2ab \cos C$$

$$c^2 = a^2 (\cos^2 C + \sin^2 C) + b^2 - 2ab \cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

We can use the Law of Cosines to solve triangles where the Law of Sines is insufficient such as the case where three sides are given, called the SSS case. If three side lengths can possibly form a triangle, then the sum of any two of the lengths must be greater than the length of the third side according to the Triangle Inequality below.

Triangle Inequality: For all real numbers a and b , we have $|a + b| \leq |a| + |b|$.

Consider a triangle with side lengths $a = 48$, $b = 36$, and $c = 31$. The sum of any two of these lengths will exceed the third; hence, a triangle is possible. However, we have not yet ruled out the possibility that more than one triangle is possible. When multiple triangles might be possible in the SSS case, we always solve for the largest angle first, which is opposite the longest side. Since a is the longest side, we will solve for the measure of angle A . To solve for the measure of $\angle A$, we simply substitute the known quantities into the Law of Cosines then solve as follows.

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc \cos A \\
(48)^2 &= (36)^2 + (31)^2 - 2(36)(31)\cos A \\
2,304 &= 1,296 + 961 - 2,232 \cos A \\
2,304 &= 2,257 - 2,232 \cos A \\
47 &= -2,232 \cos A \\
\frac{47}{-2,232} &= \cos A \\
-0.0210573477 &\approx \cos A \\
\cos^{-1}(-0.0210573477) &\approx A \\
91.2^\circ &\approx A
\end{aligned}$$

Since the largest angle is obtuse, we preclude the possibility of two triangles. Now, we can employ either the Law of Sines or the Law of Cosines to find a second angle. We will do the former.

$$\begin{aligned}
\frac{\sin(91.2^\circ)}{48} &= \frac{\sin C}{32} \\
32 \cdot \frac{\sin(91.2^\circ)}{48} &= \sin C \\
0.6665204556 &\approx \sin C \\
\sin^{-1}(0.6665204556) &\approx C \\
41.8^\circ &\approx C
\end{aligned}$$

To find the final angle, we can simply subtract the measures of angles A and C from 180 degrees.

Interestingly, we can use the Law of Cosines to produce the Full Pythagorean Theorem stated below.

Full Pythagorean Theorem: Consider $\triangle ABC$. If $\angle C$ is a right angle, then $c^2 = a^2 + b^2$. If $\angle C$ is obtuse, then $c^2 < a^2 + b^2$. If $\angle C$ is acute, then $c^2 > a^2 + b^2$.

This extension of the Pythagorean Theorem seems obvious when we consider the Law of Cosines, $c^2 = a^2 + b^2 - 2ab \cos C$, and that if $\angle C$ is acute, $\cos C$ is positive, but if $\angle C$ is obtuse, $\cos C$ is negative.

Suggested Homework

Section 7.2: #1-13 odd

Application Exercise

An engineer wants to position three pipes at the vertices of a triangle, as shown in the figure below. If the pipes, A , B , and C have radii 2 inches, 3 inches, and 4 inches, respectively, then what are the measures of the angles of triangle ABC .

