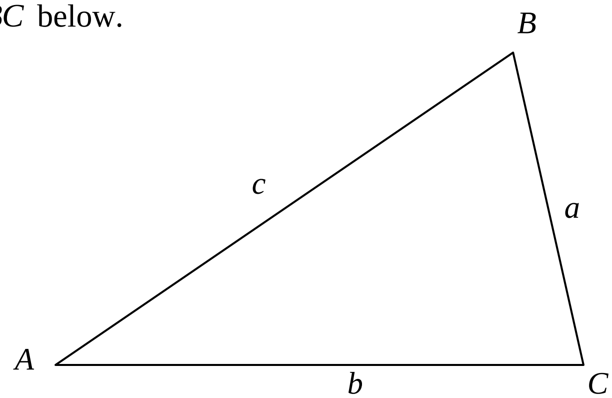
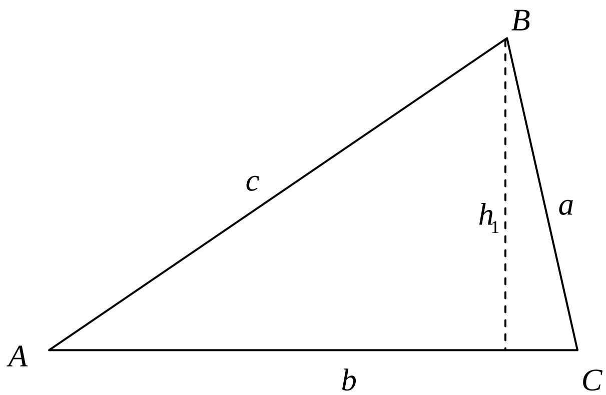
Law of Sines

The *Law of Sines* says that the ratio of the sine of an angle and the length of the side opposite the angle is the same for each angle of a triangle. For proof, we consider the case of an acute triangle here and leave the case of the obtuse triangle as an exercise.

Consider $\triangle ABC$ below.




We begin by constructing an altitude $h_1$ from vertex $B$ to side $b$ as shown.

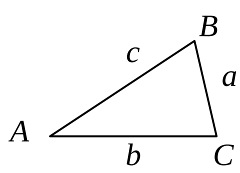



Now, we note $\sin A = h_1/c$, so $h_1 = c\sin A$. Likewise, $\sin C = h_1/a$, so $h_1 = a\sin C$. Hence, we have $c\sin A = a\sin C$, which implies the following.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Constructing another altitude leads to the proof's conclusion. Hence, we have the Law of Sines stated below.

The *Law of Sines* states that for any $\triangle ABC$ acute, right, or obtuse, then




$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We can also write the Law of Sines as below.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We use the Law of Sines to solve triangles, that is, to solve for some unknown measure of a triangle be it an angle or side. If we are given the measure of two angles and one side length, solving for the remaining unknown quantities is straightforward. For instance, imagine that $\angle A$ measures 28º, $\angle B$ measures 66º, and the side oppose $\angle C$ measures 8.2 units. Since the three angles of a triangle are supplementary, we know $\angle C$ measures $86°$. By the Law of Sines, we have the following.

$$\frac{8.2}{\sin 86°} = \frac{b}{\sin 66°}$$
$$b = \sin 66° \cdot \frac{8.2}{\sin 86°}$$
$$b \approx 7.5$$

Again, by Law of Sines, we finish below.

$$\frac{7.5}{\sin 66°} = \frac{a}{\sin 28°}$$
$$a = \sin 28° \cdot \frac{7.5}{\sin 66°}$$
$$a \approx 3.9$$

Hence, the three angles measure 28, 66, and 86 degrees, and the three sides measure 8.2, 7.5, and 3.9 units in length.

If only one angle is given, then two sides are needed. If the given angle is opposite from one of the two given sides, we have what is called the ambiguous case since more than one possible triangle (or no triangle) is possible. Assume that the measure of $\angle A$ is known as well as sides $b$ and $a$. Then, the diagram below demonstrates the ambiguity.

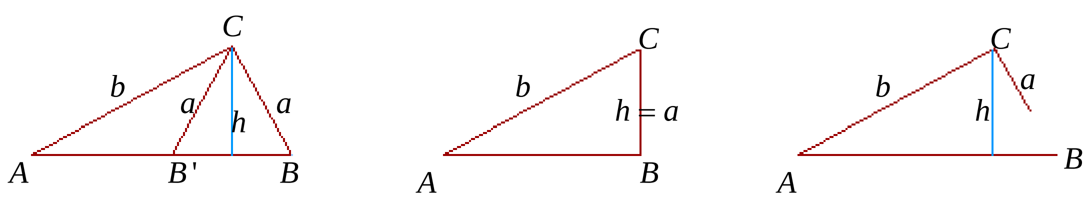



Fig. 1 Fig. 2 Fig. 3

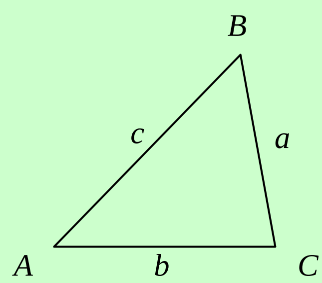
To determine the case, solve normally. If the given angle is obtuse, there is either one solution or no solution both are readily apparent. If the given angle is acute, then if there is only one solution it is either a right triangle or isosceles. In the case, where there exists a solution, but the triangle is not a right triangle nor isosceles, then two solutions exist. Figure 1 above illustrates this case. To solve for the second possible triangle, we need only know that $\angle B$ and $\angle B'$ will be supplementary.

Suggested Homework

Section 7.1: #1-17 odd

Application Exercise

The formulas below calculate the area $A$ of $\triangle ABC$.




$$A = \frac{1}{2} bc \sin A$$

$$A = \frac{1}{2} ac \sin B$$

$$A = \frac{1}{2} ab \sin C$$

Show that a formula for the area $K$ of parallelogram $\square ABCD$ is $K = bd \sin A$.