

## Conditional Identities

Consider the equation,  $2 \sin x - 1 = 0$ . We call equations like this conditional identities because they are true only on the condition that  $x$  takes certain values. In this lecture, we learn to solve for these conditional  $x$ -values.

Let's consider the given equation,  $2 \sin x - 1 = 0$ . Isolating the trigonometric expression, we obtain,  $\sin x = 1/2$ ; hence,  $x = \pi/6$  and  $x = 5\pi/6$  if we restrict  $x$  to values less than  $2\pi$ . Abdicating this restriction, the solutions reoccur every cycle of sine's period be it in the positive or negative direction. Consequently, we can say,  $2 \sin x - 1 = 0$  if  $x = \pi/6 + 2k\pi$  and  $x = 5\pi/6 + 2k\pi$  where  $k$  is an integer.

To understand our general condition above, let's look at the solution  $x = \pi/6$ . It is easy to see that the identity is true again if  $x = 13\pi/6$  (note that  $\pi/6$  and  $13\pi/6$  are co-terminal angles). Consider the expression  $x = \pi/6 + 2k\pi$  where  $k = 1$ :

$$\begin{aligned}x &= \frac{\pi}{6} + 2k\pi \\x &= \frac{\pi}{6} + 2(1)\pi \\x &= \frac{\pi}{6} + 2\pi \\x &= \frac{\pi}{6} + \frac{12\pi}{6} \\x &= \frac{13\pi}{6}\end{aligned}$$

In the example above, we isolated the trigonometric expression. Another strategy requires factoring. Consider the conditional identity:  $\cot x \cos^2 x = 2 \cot x$ . To solve for the conditional values of  $x$ , we will set one side equal to zero and factor as below.

$$\begin{aligned}\cot x \cos^2 x &= 2 \cot x \\ \cot x \cos^2 x - 2 \cot x &= 0 \\ \cot x (\cos^2 - 2) &= 0\end{aligned}$$

We have a product equal to zero; hence, one of the factors must equal zero for the condition to be true. Setting each factor equal to zero, we obtain solutions.

$$\begin{aligned}\cot x &= 0 & \cos^2 - 2 &= 0 \\ x &= \frac{\pi}{2} + k\pi & \cos x &\neq \pm\sqrt{2}\end{aligned}$$

Note that one of the factors could not equal zero ( $\pm\sqrt{2}$  falls outside the range of  $\cos x$ ).

Conditional identities can take the form of a quadratic as below.

$$2 \sin^2 x - 1 = \sin x$$

In which case, we apply strategies similar to a quadratic equation, i.e., we factor if possible and otherwise rely on the quadratic formula.

$$\begin{aligned}
 2\sin^2 x - 1 &= \sin x \\
 2\sin^2 x - \sin x - 1 &= 0 \\
 (2\sin x + 1)(\sin x - 1) &= 0 \\
 2\sin x + 1 = 0, \quad \sin x - 1 &= 0 \\
 \sin x = -1/2, \quad \sin x = 1 & \\
 x = \frac{7\pi}{6} + 2k\pi, \quad x = \frac{\pi}{2} + 2k\pi & \\
 x = \frac{11\pi}{6} + 2k\pi &
 \end{aligned}$$

Sometimes using non-conditional trigonometric identities help solve a conditional identity as below.

$$\begin{aligned}
 2\sin^2 x + 3\cos x &= 3 \\
 2\sin^2 x + 3\cos x - 3 & \\
 2(1 - \cos^2 x) + 3\cos x - 3 & \\
 2 - 2\cos^2 x + 3\cos x - 3 &= 0 \\
 -2\cos^2 x + 3\cos x - 1 &= 0 \\
 2\cos^2 x - 3\cos x + 1 &= 0 \\
 (2\cos x - 1)(\cos x - 1) &= 0 \\
 2\cos x - 1 = 0, \quad \cos x - 1 &= 0 \\
 \cos x = 1/2, \quad \cos x = 1 & \\
 x = \frac{\pi}{3} + 2k\pi, \quad x = 2k\pi & \\
 x = \frac{5\pi}{3} + 2k\pi &
 \end{aligned}$$

Sometimes squaring both sides offers an avenue towards a solution, but we must remember that squaring both sides of an equation can introduce extraneous roots.

$$\begin{aligned}
 \cos x + 1 &= \sin x \\
 (\cos x + 1)^2 &= (\sin x)^2 \\
 \cos^2 x + 2\cos x + 1 &= \sin^2 x \\
 \cos^2 x + 2\cos x + 1 &= 1 - \cos^2 x \\
 2\cos^2 x + 2\cos x &= 0 \\
 (\cos x + 1) \cdot 2\cos x &= 0 \\
 \cos x = -1 \quad \text{or} \quad \cos x = 0 & \\
 x = \pi \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2} &
 \end{aligned}$$

Checking the solutions shows that  $3\pi/2$  is extraneous, so the solutions are  $x = \pi + 2\pi k$  and  $x = \pi/2 + 2\pi k$ .

Some equations involve multiple angles. Start by solving for the multiple angle.

$$2 \cos(3x) - 1 = 0$$

$$2 \cos(3x) = 1$$

$$\cos(3x) = \frac{1}{2}$$

Concentrating on the argument,  $3x$ , we see:

$$3x = \frac{\pi}{3} + 2\pi k \quad \text{and} \quad 3x = \frac{5\pi}{3} + 2\pi k$$

Dividing by the multiple, we obtain the solutions.

$$x = \frac{\pi}{9} + \frac{2\pi k}{3} \quad \text{and} \quad x = \frac{5\pi}{9} + \frac{2\pi k}{3}.$$

All the previous examples involve common angles (or some simple fraction of a common angle). The solutions in the following conditional identity are not nice, convenient angles.

$$\sec^2 x - 2 \tan x = 4$$

$$1 + \tan^2 x - 2 \tan x = 4$$

$$\tan^2 x - 2 \tan x - 3 = 0$$

$$(\tan x - 3)(\tan x + 1) = 0$$

$$\tan x - 3 = 0, \quad \tan x + 1 = 0$$

$$\tan x = 3, \quad \tan x = -1$$

Here we come to an impasse, until we realize that we can undo a trigonometric function by its inverse as demonstrated below.

$$\arctan(\tan x) = \arctan(3)$$

$$x = \arctan(3)$$

$$x = \arctan(3) + \pi k$$

$$\arctan(\tan x) = \arctan(-1)$$

$$x = \arctan(-1)$$

$$x = \arctan(-1) + \pi k$$

Use a calculator to acquire approximations for the  $x$ -values above.

## Suggested Homework

Section 6.6: #1-17 odd, #31-45 odd, #63-89 odd

### Application Exercise

The distance  $d$  traveled by a projectile fired at an angle  $\theta$  is related to the initial velocity  $v_0$  (in feet per second) by the equation  $v_0^2 \sin 2\theta = 32d$ . If the muzzle velocity for a rifle equals 230 feet per second, then at what angle would it have to be aimed for the bullet to travel 826.5625 feet?