

## Product to Sum Identities

In this lecture, we will acquire a new identities from the sum and difference identity. We add  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$  to  $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$  as below.

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin \alpha \cos \beta + \sin \alpha \cos \beta + \cos \alpha \sin \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

$$\frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} = \sin \alpha \cos \beta$$

Similarly, we produce the Product-to-Sum Identities below.

Product-to-Sum Identities:

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

$$\cos \alpha \sin \beta = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$$

$$\cos \alpha \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

Next, we let  $\alpha + \beta = x$  and  $\alpha - \beta = y$ . Adding these equations, we get  $x + y = 2\alpha$ .

Subtracting the two equations, gives us  $x - y = 2\beta$ . Thus, we have  $\alpha = \frac{x+y}{2}$  and  $\beta = \frac{x-y}{2}$ ,

which we substitute into  $\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$  as below.

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{\sin\left(\frac{x+y}{2} + \frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2} - \frac{x-y}{2}\right)}{2}$$

$$2 \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \sin(x) + \sin(y)$$

Similarly, it is possible to obtain the following Sum-to-Product Identities.

Sum-to-Product Identities:

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

## Suggested Homework

Section 6.5: #51-59 odd

### Application Exercise

Use the Sum and Difference Identities from the previous lecture to derive one of the Product-to-Sum Identities other than  $\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$ .