

## Double-Angle and Half-Angle Identities

In the last lecture, we showed  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ . In this lecture, we will acquire the so-called Double-Angle Identity. We let  $\alpha = \beta = x$  and substitute into the Sine of a Sum Identity as below.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

$$\sin(2x) = 2 \sin x \cos x$$

Hence, we have the following identity.

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

For cosine, we do the same as below.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(x + x) = \cos x \cos x - \sin x \sin x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Using the Pythagorean Identity, we attain another common identity for  $\cos 2x$  as below.

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos(2x) = \cos^2 x - (1 - \cos^2 x)$$

$$\cos(2x) = \cos^2 x - 1 + \cos^2 x$$

$$\cos(2x) = 2 \cos^2 x - 1$$

If we replace  $\cos^2 x$  using the Pythagorean Identity instead of  $\sin^2 x$ , we attain still another common identity for  $\cos 2x$ .

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos(2x) = (1 - \sin^2 x) - \sin^2 x$$

$$\cos(2x) = 1 - \sin^2 x - \sin^2 x$$

$$\cos(2x) = 1 - 2 \sin^2 x$$

So, we have three Double-Angle Identities for cosine stated below.

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

Similarly, we can attain a Double-Angle Identity for tangent.

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

Thus, we have the Double-Angle Identities stated here.

Double-Angle Identities:

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

We can use the Double-Angle Identities to acquire the following Half-Angle Identities.

Half-Angle Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

### Example Exercise 1

$$\text{Verify } \sin(2s) = -2 \sin s \cdot \sin(s - \pi/2).$$

By the Sine of a Difference Identity, we have the equality below.

$$\sin(2s) = -2 \sin s \cdot [\sin s \cdot \cos(\pi/2) - \cos s \cdot \sin(\pi/2)]$$

Evaluating sine and cosine at  $\pi/2$ , we obtain the following.

$$\sin(2s) = -2 \sin s \cdot [-\cos s]$$

By the commutative property of multiplication, we arrive at the Double Angle Identity for Sine.

$$\sin(2s) = -2 \cdot -1 \cdot \sin s \cdot \cos s$$

$$\sin(2s) = 2 \sin s \cdot \cos s$$

### Example Exercise 2

$$\text{Verify } \frac{\cos 2x}{\cos^2 x} = \sec^2 x - 2 \tan^2 x.$$

First, we apply the Double-Angle Identity and decompose the resulting fractions as follows.

$$\frac{\cos^2 x - \sin^2 x}{\cos^2 x} = \sec^2 x - 2 \tan^2 x$$

$$\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \sec^2 x - 2 \tan^2 x$$

Second, we apply the Pythagorean Identity and decompose yet again as below.

$$\frac{1 - \sin^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \sec^2 x - 2 \tan^2 x$$

$$\frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = \sec^2 x - 2 \tan^2 x$$

Finally, we apply the Reciprocal Identity and Fundamental Identity, and we add.

$$\sec^2 x - \tan^2 x - \tan^2 x = \sec^2 x - 2 \tan^2 x$$

$$\sec^2 x - 2 \tan^2 x = \sec^2 x - 2 \tan^2 x$$

## Suggested Homework

Section 6.4: #53-65 odd

### Application Exercise

Physicists often approximate the acceleration due to gravity,  $g$ , as 9.8 meters per second squared. However,  $g$  varies slightly with latitude. If  $\theta$  is the measure of latitude in degrees, then the following formula approximates  $g$  in meters per second squared.

$$g \approx 9.78049(1 + 0.005288 \sin^2 \theta - 0.000006 \sin^2 2\theta)$$

Write an expression for  $g$  in terms of  $\sin \theta$  only.