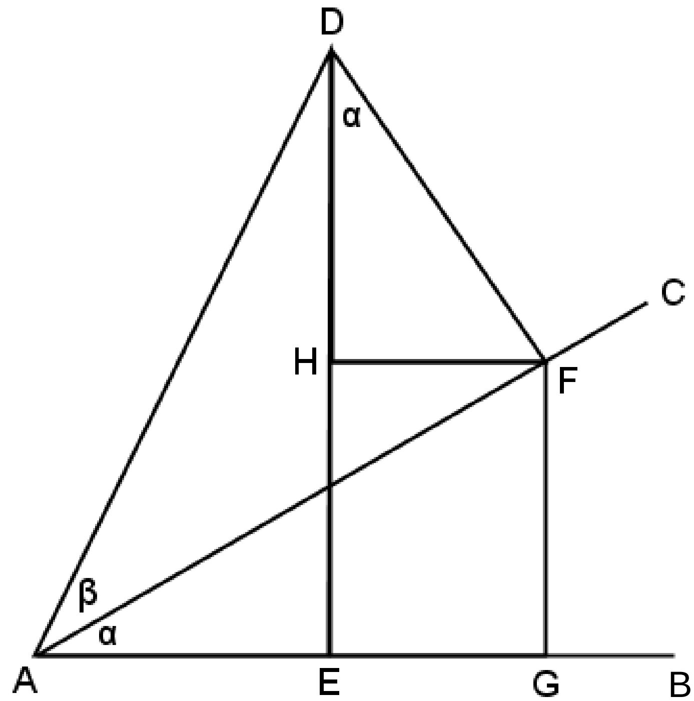
Sum and Difference Identities

In the last lecture, we learned how to use basic identities to verify other interrelationships between the trigonometric functions. In this lecture, we will append some additional identities called the Sum and Difference Identities to our list of basic facts that we can use to verify further interrelationships.

We want to establish an identity for the sine of the sum of two angles α and β. Consider $\overleftrightarrow{AB}$ in the diagram below. Let $\overleftrightarrow{AB}$ revolve to point *C* and sweep out angle $\angle CAB$, measuring α, then revolve further to point *D* and sweep out angle $\angle CAD$, measuring β, as shown.



From point *D*, we draw $\overline{DE}$ perpendicular to $\overleftrightarrow{AB}$. Then, $\sin(\alpha+\beta)=\overline{ED}/\overline{DA}$, and $\cos(\alpha+\beta)=\overline{AE}/\overline{DA}$.

Next, we draw $\overline{DF}$ perpendicular to $\overline{AC}$, $\overline{FG}$ perpendicular to $\overleftrightarrow{AB}$, and $\overline{FH}$ perpendicular to $\overline{ED}$. This makes $\overline{AC}$ a transversal crossing the parallels $\overleftrightarrow{AB}$ and $\overline{HF}$. Hence, $\angle HFA$ and $\angle CAB$ are alternate interior angles and congruent. Moreover, $\angle HDF$ and $\angle HFA$ are also congruent because they are both complementary to $\angle HFD$. We have $\angle CAB \cong \angle HFA$ and $\angle HDF \cong \angle HFA$. By the transitive property, $\angle CAB \cong \angle HDF$. Thus, $m(\angle HDF)=\alpha$.

Now we note $\overline{ED}=\overline{GF}+\overline{HD}$, and we substitute as below.

$$\sin(\alpha+\beta)=\frac{\overline{ED}}{\overline{DA}}=\frac{\overline{GF}+\overline{HD}}{\overline{DA}}=\frac{\overline{GF}}{\overline{DA}}+\frac{\overline{HD}}{\overline{DA}}$$

Multiplying by a judicious form of 1, we obtain the following.

$$\sin(\alpha+\beta)=\frac{\overline{GF}}{\overline{DA}}+\frac{\overline{HD}}{\overline{DA}}$$
$$\sin(\alpha+\beta)=\frac{\overline{GF}}{\overline{DA}}\cdot\frac{\overline{AF}}{\overline{AF}}+\frac{\overline{HD}}{\overline{DA}}\cdot\frac{\overline{FD}}{\overline{FD}}$$
$$\sin(\alpha+\beta)=\frac{\overline{GF}}{\overline{AF}}\cdot\frac{\overline{AF}}{\overline{DA}}+\frac{\overline{HD}}{\overline{FD}}\cdot\frac{\overline{FD}}{\overline{DA}}$$

Now, we note the following equalities by definition, $\sin(\alpha)=\overline{GF}/\overline{AF}$, $\cos(\beta)=\overline{AF}/\overline{DA}$, $\cos(\alpha)=\overline{HD}/\overline{FD}$, and $\sin(\beta)=\overline{FD}/\overline{DA}$. We conclude by substitution.

$$\sin(\alpha+\beta)=\frac{\overline{GF}}{\overline{AF}}\cdot\frac{\overline{AF}}{\overline{DA}}+\frac{\overline{HD}}{\overline{FD}}\cdot\frac{\overline{FD}}{\overline{DA}}$$
$$\sin(\alpha+\beta)=\sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$$

Hence, we have the following identity.

Sine of a Sum Identity:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

The remaining Sum and Difference Identities we will state without proof.

Sum and Difference Identities:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example Exercise 1

Verify $\cos\left(x-\frac{\pi}{2}\right)=\cos x\tan x$.

By the Difference Identity of Cosine, we obtain the following.

$$\cos\left(x-\frac{\pi}{2}\right)=\cos x\tan x$$
$$\cos(x)\cos\left(\frac{\pi}{2}\right)+\sin(x)\sin\left(\frac{\pi}{2}\right)=\cos x\tan x$$

Evaluating $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$ yields

$$\cos(x)\cdot 0+\sin(x)\cdot 1=\cos x\tan x$$
$$\sin(x)=\cos x\tan x.$$

Multiplying by a fortuitous form of “1” yields

$$\frac{\cos x}{\cos x}\cdot\sin x=\cos x\tan x$$
$$\cos x\cdot\frac{\sin x}{\cos x}=\cos x\tan x.$$

The Fundamental Identity completes the verification.

$$\cos x\cdot\tan x=\cos x\tan x$$

Example Exercise 2

Verify $\dfrac{\cos(x-y)}{\sin(x+y)} = \dfrac{1+\tan x \tan y}{\tan x + \tan y}$.

By the Fundamental Identity, we have

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{1+\dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}.$$

Adding the fractions in the denominator, we obtain

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{1+\dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}{\dfrac{\sin x\cdot\cos y}{\cos x\cdot\cos y}+\dfrac{\sin y\cdot\cos x}{\cos y\cdot\cos x}}$$

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{1+\dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}{\dfrac{\sin x\cdot\cos y+\sin y\cdot\cos x}{\cos x\cdot\cos y}}.$$

Division, yields

$$\frac{\cos(x-y)}{\sin(x+y)} = \left(1+\frac{\sin x\cdot\sin y}{\cos x\cdot\cos y}\right)\cdot\frac{\cos x\cdot\cos y}{\sin x\cdot\cos y+\sin y\cdot\cos x}$$

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{\cos x\cdot\cos y}{\sin x\cdot\cos y+\sin y\cdot\cos x}+\frac{\sin x\cdot\sin y}{\sin x\cdot\cos y+\sin y\cdot\cos x}.$$

Adding the fractions, we obtain

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{\cos x\cdot\cos y+\sin x\cdot\sin y}{\sin x\cdot\cos y+\sin y\cdot\cos x}.$$

The Cosine of a Difference Identity and the Sine of a Sum Identity complete the verification.

$$\frac{\cos(x-y)}{\sin(x+y)} = \frac{\cos(x-y)}{\sin(x+y)}$$

Suggested Homework

Section 6.3: #11-17 odd, #79-97 odd

Application Exercise

Scientists use the following equation in the study of the electromagnetic wave theory of light.

$$E'' = -E\left[\frac{k\cos r - \cos i}{k\cos r + \cos i}\right]$$

In the equation, k is the index of refraction, $\sin i / \sin r$.

Show that the above equation is equivalent to the equation below.

$$E'' = -E\left[\frac{\sin(i-r)}{\sin(i+r)}\right]$$