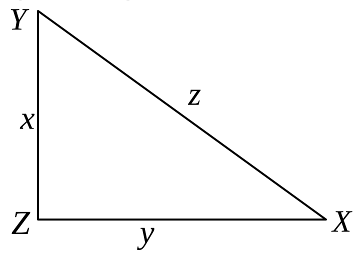
Angles

In this lecture, we review essential geometry and cover some very basic vocabulary. To start, we recall the Pythagorean Theorem, which states that for any right triangle the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse with the hypotenuse being the side opposite the right angle (the longest side). .

Let $\triangle XYZ$ be a right triangle as shown below.



Then, $x^2 + y^2 = z^2$.

In particular, a right triangle with two 45-degree angles has two legs with equal length. Hence, we acquire the 45º-45º rule below.

$$x^2 + y^2 = z^2$$
$$x^2 + x^2 = z^2$$
$$2x^2 = z^2$$
$$x^2 = \frac{z^2}{2}$$
$$x = \frac{z}{\sqrt{2}} \text{ or } \frac{z\sqrt{2}}{2}$$

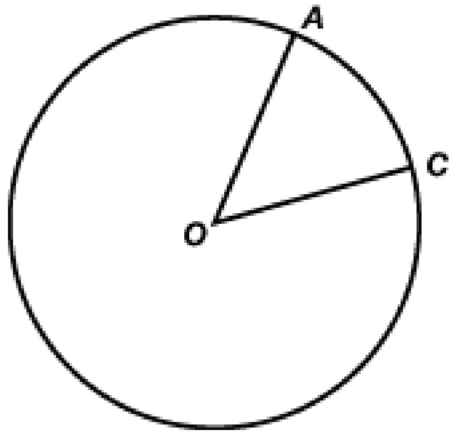
Let $\triangle XYZ$ be a right triangle. Let z represent the length of the hypotenuse. Let x and y represent the length of the legs. If one of the interior angles equals 45º, then $x = y = z/\sqrt{2}$.

Another common rule is the 30º-60º rule stated below.

Let $\triangle XYZ$ be a right triangle. Let z represent the length of the hypotenuse. Let x represent the length of the shortest side, and let y represent the length of the other leg. If one of the interior angles equals 30º, then the following equalities hold.

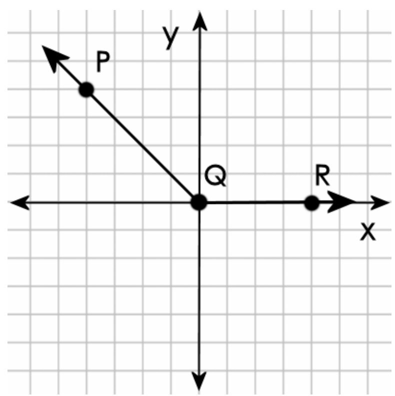
$$x = \frac{z}{2},\ y = x \cdot \sqrt{3},\ y = \frac{z \cdot \sqrt{3}}{2}$$

Now, we turn to some basic vocabulary concerning angles and their measure. A *central angle* is an angle with the center of a circle as its vertex. The angle in the figure below is a central angle.



We denote this angle as $\angle AOC$ or $\angle COA$ or simply $\angle O$. The measure of a central angle equals the measure of its intercepted arc. In other words, if a central angle cuts away $N°$ of a circle then the angle measures $N°$. Thus, the degree measure of $\angle O$ equals the degree measure of $\overset{\frown}{AC}$.

An angle in *standard position* has its initial side on the positive *x*-axis and its vertex at the origin. $\angle PQR$ pictured below is in standard position.



A *quadrantal angle* is an angle in standard position with its terminal side on the *x* or *y*-axis. *Coterminal angles* have a common terminal side when in standard position.

We commonly measure angles in degrees and radians. We have discussed degree measure above when we defined a central angle, but the box below gives the formal definition of the degree measure of an angle.

> The *degree measure of an angle* is the number of degrees in the intercepted arc of a circle centered at its vertex. The degree measure is positive if the rotation is counterclockwise and negative if the rotation is clockwise.

Using this definition, we can more formally define coterminal angles as below.

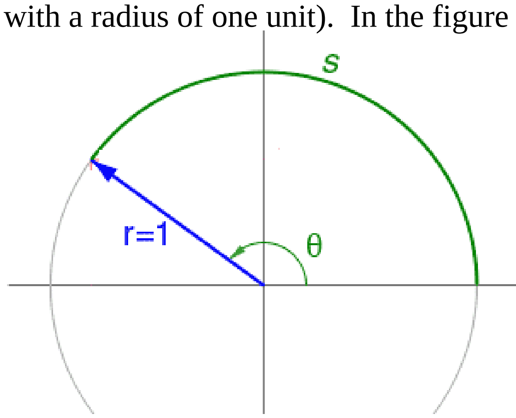
> Let angles A and B be *coterminal*. Then, $\exists\ k \in \mathbb{Z} \ni m(B) = m(A) + k \cdot 360°$

For example, consider two angles, one with degree-measure $-549°$ and one with degree-measure $171°$. Let $m(A) = -549°$ and $m(B) = 171°$, then solve for *k* in the equation below.

$$\begin{aligned}
&171° = -549° + k \cdot 360° \\
&171° + 549° = k \cdot 360° \\
&\frac{720°}{360°} = \frac{k \cdot 360°}{360°} \\
&2 = k
\end{aligned}$$

Since *k* is an integer, then the two angles are coterminal in standard position. If *k* had not been an integer, then the angles would not be coterminal in standard position.

Radian measure is another way to measure angles. The radian measure of an angle in standard position equals the length of the intercepted arc on the unit circle (the unit circle is a circle centered at the origin with a radius of one unit). In the figure below, the radian measure of angle θ is *s*.



Formally, we define radian measure as below.

> The radian measure of an angle in standard position equals the length of the intercepted arc on the unit circle if the rotation is counterclockwise; otherwise, the radian measure equals the opposite of the length of the arc.

Since the circumference of a circle is $2\pi r$, the circumference of the unit circle is 2π. Hence, if we rotate the terminal side $180°$ (one-half of a full revolution), the measure of the resulting angle is π. We use this equality for conversion from degrees to radians or from radians to degrees. For example, to convert $m = 36°$ to radians, we solve the proportion below.

$$\frac{m}{36°} = \frac{\pi}{180°}$$

$$m = \frac{\pi}{5} \approx 0.6283$$

Thus, an angle measuring 36 degrees measures $\pi/5$ radians, which is approximately 0.6283 radians.

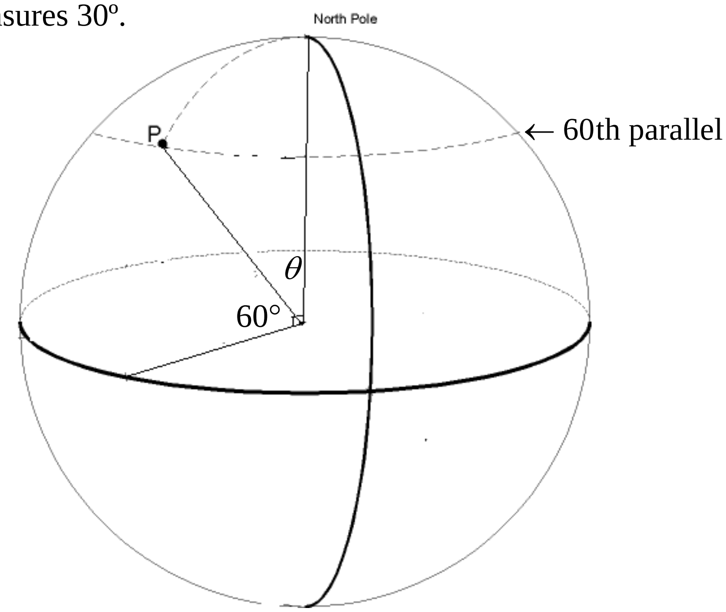
Radian measure can be used to calculate the length of an arc intercepted by a central angle. Let α represent the radian measure of a central angle inscribed in a circle with radius *r*. Let *s* represent the length of the arc intercepted by the angle. The radian measure of the central angle represents a fraction of the circumference of the circle, namely, $\alpha/2\pi$. Multiplying this fraction by the full circumference gives the arc length as below.

$$s = \frac{\alpha}{2\pi} \cdot 2\pi r$$
$$s = \alpha r$$

Hence, we have the Arc Length Theorem stated below.

> *Arc Length Theorem*: Let *s* represent the length of an arc intercepted by a central angle of α radians on a circle of radius *r*. Then, $s = \alpha r$.

Using the theorem below, we can calculate the distance from point *P* to the North Pole given the latitude of point *P*. The latitude or parallel of a point on the earth's surface in the northern hemisphere equals the complement of the measure of the central angle subtended by the arc extending from the point to the North Pole. In other words, if point *P* in the diagram below has latitude of 60º, then θ measures 30º.



If the radius of the earth is 3,950 miles, then how far is point *P* from the North Pole? To answer, convert the angle measure to radians and use the Arc Length Theorem as below.

$$s = \alpha r$$
$$s = \frac{\pi}{6} \cdot 3{,}950 \text{ miles} \approx 2{,}068 \text{ miles}$$

Example Exercise #1

Show that angles measuring $-15°$ and $1,065°$ are coterminal.

If angles A and B are coterminal, then there exists an integer k such that we have the equation below.

$$m(\angle B) = m(\angle A) + k360°$$

Substituting our angle measures, we solve for k and show that it is an integer.

$$\begin{aligned}
&1,065° = -15° + k \cdot 360° \\
&1,080° = k \cdot 360° \\
&\frac{1,080°}{360°} = k \\
&3 = k
\end{aligned}$$

Since $k \in \mathbb{Z}$, the angles measuring $-15°$ and $1,065°$ are coterminal.

Example Exercise #2

Convert $12°$ to radian measure.

Recall that $\pi = 180°$. Write and solve the proportion.

$$\begin{aligned}
&\frac{x}{12°} = \frac{\pi}{180°} \\
&x = 12° \cdot \frac{\pi}{180°} \\
&x = \cancel{12°} \cdot \frac{\pi}{\underset{15}{\cancel{180°}}} \\
&x = \frac{\pi}{15} \approx 0.21
\end{aligned}$$

Suggested Homework

Section 5.1: #1-3 odd, #23-27 odd, #55-79 odd, #91-101 odd

Application Exercise

If a point is in motion on a circle of radius r through an angle of α radians in time t, then its *linear velocity* is $v = s/t$ where s is the arc length and its *angular velocity* is $\omega = \alpha/t$. Use these definitions as well as the *Arc Length Theorem* to prove the theorem below:

If v is the linear velocity of a point on a circle of radius r and ω is its angular velocity then $v = r\omega$.